

1. **(Writing Exponential Functions)** The number of native speakers of an ancient language has been decreasing exponentially since 2000.

(a) Assume that there were 200 native speakers in 2005 and 160 native speakers in 2010. Write an exponential function $P(t)$ that models the number of native speakers t years after 2000.

(b) Assume instead that there were 250 native speakers in 2000. And when there were 200 native speakers, the number was decreasing at a rate of 10 people per year. Write an exponential function $Q(t)$ that models the situation.

2. **(Derivative of Exponentials and Logarithms)** Differentiate each function below.

(a) $f(x) = 2^x + \log_2 x + \ln 2$

(b) $f(x) = \frac{e^{-2x}}{x+1}$

(c) $f(x) = x^3 \ln(5x^4)$

(d) $f(x) = \log_7(3x\sqrt{x})$

All the skills you should know about exponentials and logarithms were contained in the worksheet for Lecture 34. Make sure you understand each subproblem!

3. **(Linear Approximation)**

(a) Use tangent line approximation to estimate $\frac{1}{\sqrt{5}}$. Is it an overestimate or underestimate?

(b) Use a secant line approximation to get an overestimate of $\frac{1}{\sqrt{5}}$.

(c) Use another secant line to get an underestimate of $\frac{1}{\sqrt{5}}$.

4. **(Analysis of Extrema)** Let $f(x) = -x^3 - 3x^2 + 9x + 1$.

(a) Sketch the graph of $f(x)$, labeling local extrema and inflection points.

(b) Write down the global extrema of $f(x)$ on $(-4, 3)$ if they exist.

(c) Write down the global extrema of $f(x)$ on $[-2, 2]$ if they exist.

5. **(Optimization)** It just snowed 50π cubic feet of snow in Jack's back yard. Jack is planning on using this to build a snowman and a snowdog. The snowman consists of balls of snow of radii x , $2x$ and $3x$ stacked on top of each other for some length x (measured in feet). The snowdog consists of a cylinder of snow of radius 1 foot and length ℓ feet, for some ℓ .
- (a) Given the limited amount of snow in the yard, write a constraint that x and ℓ must satisfy in order for Jack to actually be able to build these snowcreatures (recall that the volume of a ball of radius r is $\frac{4}{3}\pi r^3$ and the volume of a cylinder of radius r and height h is $\pi r^2 h$).
- (b) Given the constraint you found in part (a), what are the possible values of the length x ?
- (c) The judges for the neighborhood snowcreature competition award points based on the sizes of snowcreatures produced. They award 12 points for every foot of height of a snowman, and 1 point for every foot of length of a snowdog. What is the greatest number of points that Jack could achieve for this competition?

Make use of the practice exams, study sessions and office hours.

Good luck on the Final Exam!!!

Final Review – Solutions

1. (a) Since the ratio is $\frac{160}{200}$ over 5 years, $P(t)$ should be in the form

$$P(t) = P_0 \left(\frac{160}{200} \right)^{\frac{t}{5}} = P_0 \left(\frac{4}{5} \right)^{\frac{t}{5}}.$$

As we are given $P(5) = 200$, we may solve for P_0

$$\begin{aligned} P(5) &= P_0 \cdot \frac{4}{5} = 200 \\ P_0 &= 250. \end{aligned}$$

- (b) The problem tells us that at some unknown time t_0 ,

$$\frac{Q'(t_0)}{Q(t_0)} = \frac{-10}{200} = -\frac{1}{20}.$$

Because for an exponential function, the derivative and the original function are always proportional, we have just found the constant of proportionality $Q'(t)/Q(t)$ is $-\frac{1}{20}$. Hence

$$Q(t) = Q_0 e^{-\frac{t}{20}}.$$

As there were 250 native speakers in 2000, $Q_0 = 250$.

2. (a) $f'(x) = \ln 2 \cdot 2^x + \frac{1}{\ln 2 \cdot x}$.
(b)

$$\begin{aligned} f'(x) &= \frac{-2e^{-2x}(x+1) - e^{-2x} \cdot 1}{(x+1)^2} \\ &= \frac{-2e^{-2x} \cdot x - 3e^{-2x}}{(x+1)^2}. \end{aligned}$$

- (c) We first rewrite the original function

$$f(x) = x^3 \ln(5x^4) = x^3(\ln 5 + 4 \ln x).$$

Then

$$\begin{aligned} f'(x) &= 3x^2 \cdot (\ln 5 + 4 \ln x) + x^3 \cdot \frac{4}{x} \\ &= x^2(3 \ln 5 + 12 \ln x + 4). \end{aligned}$$

- (d) Again the original function can be rewritten as

$$f(x) = \log_7(3x\sqrt{x}) = \log_7 3 + \frac{3}{2} \log_7 x.$$

Then

$$f'(x) = \frac{3}{2} \cdot \frac{1}{\ln 7 \cdot x}.$$

3. The function $f(x) = \frac{1}{\sqrt{x}}$ is the one we want to do linear approximation with.

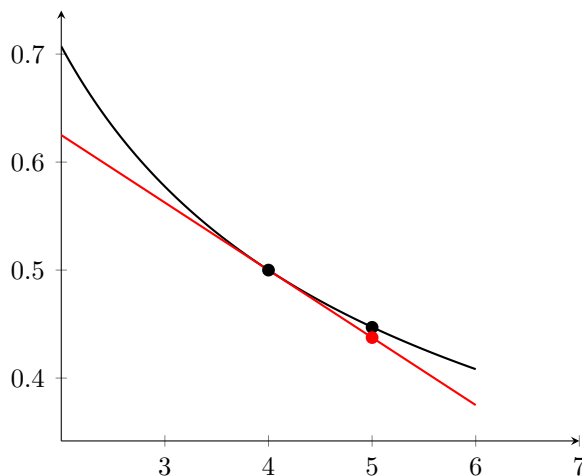
- (a) To use tangent line approximation, we choose to use the tangent line at $x = 4$. First of all we need to find the slope of the tangent line, $f'(4)$. We have $f'(x) = -\frac{1}{2\sqrt{x^3}}$, and thus $f'(4) = -\frac{1}{2 \cdot 8} = -\frac{1}{16}$. The equation of the tangent line to $y = f(x)$ at $x = 4, y = f(4) = \frac{1}{2}$ is

$$y - \frac{1}{2} = -\frac{1}{16}(x - 4).$$

Plugging in $x = 5$, we get the estimate

$$\frac{1}{\sqrt{5}} \approx \frac{1}{2} - \frac{1}{16}(5 - 4) = \frac{1}{2} - \frac{1}{16} = \frac{7}{16}.$$

This is an overestimate as the graph of $f(x)$ is concave up.



- (b) To get an overestimate for a concave up graph, we need to choose a secant line determined by two points on each side of 5. Let us choose $(4, \frac{1}{2})$ and $(9, \frac{1}{3})$. The slope of the secant line is

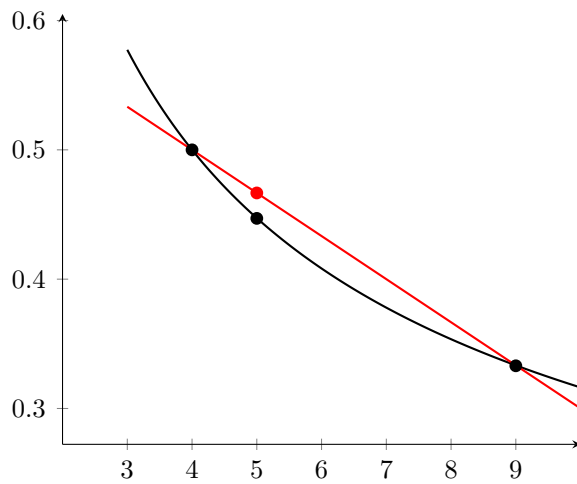
$$\frac{\frac{1}{2} - \frac{1}{3}}{4 - 9} = \frac{\frac{1}{6}}{-5} = -\frac{1}{30}.$$

Hence the equation of the secant line is

$$y - \frac{1}{2} = -\frac{1}{30}(x - 4).$$

Plugging in $x = 5$, we get the estimate

$$\frac{1}{\sqrt{5}} \approx \frac{1}{2} - \frac{1}{30}(5 - 4) = \frac{1}{2} - \frac{1}{30} = \frac{14}{30} = \frac{7}{15}.$$



- (c) To get an underestimate for a concave up graph, we need to choose a secant line determined by two points on the same side of 5. Let us choose $(1, 1)$ and $(4, \frac{1}{2})$. The slope of the secant line is

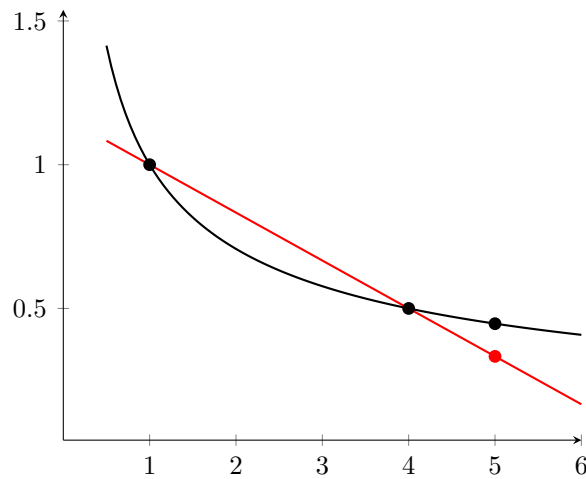
$$\frac{\frac{1}{2} - 1}{4 - 1} = \frac{-\frac{1}{2}}{3} = -\frac{1}{6}.$$

Hence the equation of the secant line is

$$y - \frac{1}{2} = -\frac{1}{6}(x - 4).$$

Plugging in $x = 5$, we get the estimate

$$\frac{1}{\sqrt{5}} \approx \frac{1}{2} - \frac{1}{6}(5 - 4) = \frac{1}{2} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$



4. (a) We have $f(x) = -x^3 - 3x^2 + 9x + 1$. We compute the derivative

$$f'(x) = -3x^2 - 6x + 9 = -3(x^2 + 2x - 3) = -3(x + 3)(x - 1)$$

with sign chart

$$\begin{array}{c|ccc} x & -3 & 1 & \\ \hline f'(x) & - & 0 & + & 0 & - \end{array}$$

Hence there is a local minimum at $x = -3$ with $f(-3) = 27 - 27 - 27 + 1 = -26$, and a local maximum at $x = 1$ with $f(1) = -1 - 3 + 9 + 1 = 6$.

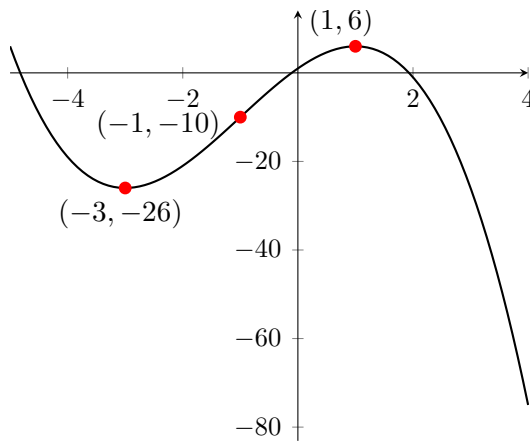
Next we compute the second derivative

$$f''(x) = -6x - 6 = -6(x + 1)$$

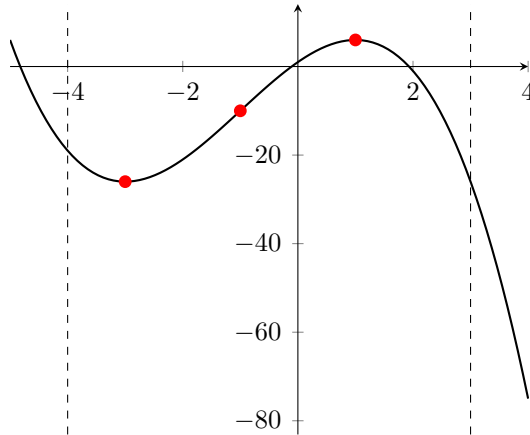
with sign chart

$$\begin{array}{c|c} x & -1 \\ \hline f''(x) & + & 0 & - \end{array}$$

Hence the graph is concave up for $x < -1$ and concave down for $x > -1$. The inflection point is at $x = -1$ with $f(-1) = 1 - 3 - 9 + 1 = -10$. The following is the sketch of $y = f(x)$.



(b) If we restrict the domain to $(-4, 3)$, we have the following graph



We need to compare $f(-4)$ and $f(1)$ to determine the global maximum.

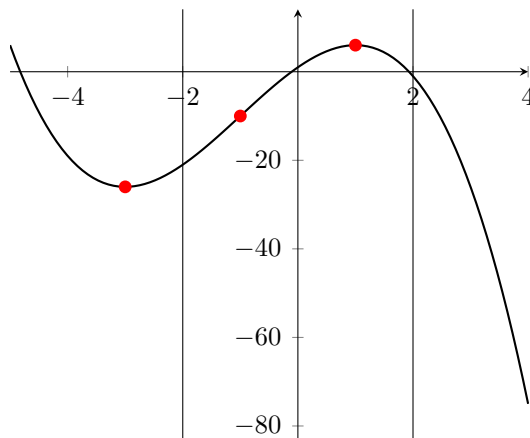
$$f(-4) = 64 - 48 - 36 + 1 = -19 < 6 = f(1),$$

and hence $x = 1$ is the global maximum. For the global minimum we need to compare $f(-3)$ and $f(3)$.

$$f(3) = -27 - 27 + 27 + 1 = -26 = f(-3),$$

so $x = -3$ is the global minimum.

(c) If we restrict the domain to $[-2, 2]$, we have the following graph



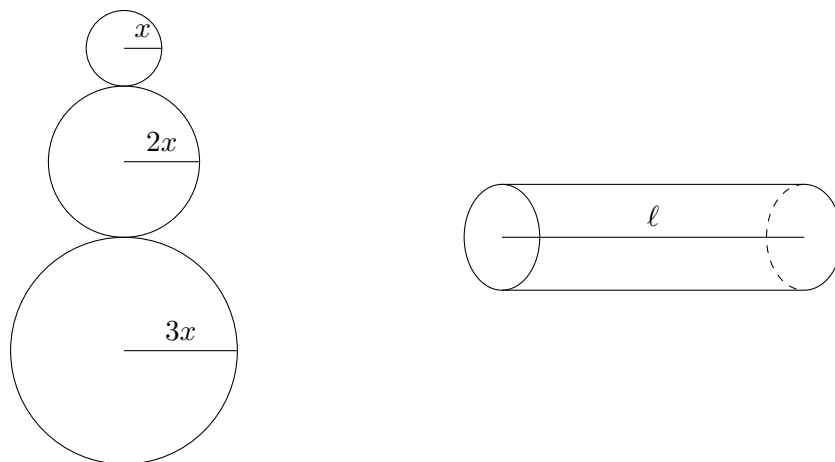
The global maximum is at $x = 1$. For the global minimum we need to compare $f(-2)$ and $f(2)$.

$$f(-2) = 8 - 12 - 18 + 1 = -21$$

$$f(2) = -8 - 12 + 18 + 1 = -1,$$

so the global minimum is at $x = -2$.

5. (a) Here is a picture to understand the problem.



The amount of snow is

$$\begin{aligned}
 50\pi &= \frac{4}{3}\pi x^3 + \frac{4}{3}\pi(2x)^3 + \frac{4}{3}\pi(3x)^3 + \pi \cdot 1^2 \cdot \ell \\
 &= \frac{4}{3}\pi(x^3 + 8x^3 + 27x^3) + \pi\ell \\
 &= \frac{4}{3}\pi \cdot 36x^3 + \pi\ell \\
 &= 48\pi x^3 + \pi\ell.
 \end{aligned}$$

Hence we have the equation

$$50 = 48x^3 + \ell,$$

or

$$\ell = 50 - 48x^3$$

- (b) Since the radius must be positive, we have

$$x > 0.$$

On the other hand, the length of the snowdog ℓ should also be positive, so

$$\ell = 50 - 48x^3 > 0,$$

or

$$x < \sqrt[3]{\frac{50}{48}} = \sqrt[3]{\frac{25}{24}}.$$

In conclusion we have

$$0 < x < \sqrt[3]{\frac{25}{24}}.$$

(c) The number of points for the competition is

$$\begin{aligned} & 12 \cdot \text{height of snowman} + 1 \cdot \text{length of snowdog} \\ &= 12 \cdot (x + x + 2x + 2x + 3x + 3x) + 1 \cdot \ell \\ &= 12 \cdot 12x + (50 - 48x^3) \\ &= -48x^3 + 144x + 50. \end{aligned}$$

Let $f(x) = -48x^3 + 144x + 50$. We want to find the maximum of $f(x)$. The derivative is

$$f'(x) = -144x^2 + 144 = -144(x+1)(x-1)$$

with sign chart

x	0	1	$\sqrt[3]{\frac{25}{24}}$
$f'(x)$	+	+	0
	-	-	-

Hence the maximum of $f(x)$ occurs at $x = 1$, with $f(1) = -48 + 144 + 50 = 146$.
Namely the greatest number of points Jack could achieve is 146.