More Exponentials and Logarithms

1. Chi-Yun was thinking hard to make problems for the final exam and suddenly she felt a headache. She took two Tylenol tablets, which contain a total of 500 mg acetaminophen. After 90 minutes, the amount of acetaminophen in her body dissipated to 400 mg.

   (a) Write down an exponential function $A(t)$, modeling the amount of acetaminophen in Chi-Yun's body t minutes after she took the tablets.

   (b) What is the rate of change of the amount of acetaminophen when Chi-Yun just took the Tylenol?

   (c) What is the percentage change of the amount of acetaminophen per minute?

   (d) How can we get an approximated numerical value for the answer in (b)? (Hint: Use linear approximation of the function __________ at $x =$ _____ to estimate the value __________.)

(e) It is known that the amount of acetaminophen needed to receive any therapeutic effect is 200 mg. At what time will the therapeutic effect wear off?

(f) If we know $\ln 2 \approx 0.69, \ln 5 \approx 1.61$, give an approximated value for your answer in (e).

(g) What is the half-life time of acetaminophen?

(h) Again use the approximated values $\ln 2 \approx 0.69, \ln 5 \approx 1.61$ to estimate your answer in (g).

(i) Sketch the graph of $A(t)$. Is $A(t)$ invertible?

(j) Find the inverse function $A^{-1}(x)$ and sketch its graph. What is the meaning of $A^{-1}(x)$?

(k) Calculate the derivative of $A^{-1}(x)$.

(l) Interpret $(A^{-1})'(200)$ in words.

More Exponentials and Logarithms – Solutions

1. (a)

$$A(t) = 500 \cdot \left(\frac{4}{5}\right)^{\frac{t}{90}}$$

(b) The rate of change of the amount of acetaminophen when Chi-Yun just took the tablets is $A'(0)$. First we calculate the derivative

$$A'(t) = 500 \cdot \left(\frac{4}{5}\right)^{\frac{t}{90}} \ln\left(\frac{4}{5}\right)^{\frac{t}{90}} = 500 \cdot \left(\frac{4}{5}\right)^{\frac{t}{90}} \cdot \frac{1}{90} \ln\left(\frac{4}{5}\right),$$

so

$$A'(0) = 500 \cdot \frac{1}{90} \cdot \ln\frac{4}{5} = \frac{50 \ln \frac{4}{5}}{9}.$$

(c) We have the formula

$$\text{Percentage Change} = \frac{\text{Final} - \text{Initial}}{\text{Initial}} \times 100\%.$$

Hence the percentage change of the amount of acetaminophen per minute is

$$\frac{500 \cdot \left(\frac{4}{5}\right)^{\frac{t}{90}} - 500}{500} \times 100\% = \left[\left(\frac{4}{5}\right)^{\frac{t}{90}} - 1\right] \times 100\%.$$

(d) We want to estimate $\ln\frac{4}{5}$ using the function $f(x) = \ln x$ at $x = 1$. The derivative $f'(x) = \frac{1}{x}$, so $f'(1) = 1$, namely the slope of the tangent line at $x = 1$ is 1. Hence the equation of the tangent line at $x = 1$ is

$$y = x - 1.$$

Consequently $\ln\frac{4}{5} \approx \frac{4}{5} - 1 = -0.2$. Our answer in (b) was

$$A'(t) = \frac{50}{9} \ln\frac{4}{5} \approx -\frac{10}{9}$$

(e) We need to solve the equation $A(t) = 500 \cdot \left(\frac{4}{5}\right)^{\frac{t}{90}} = 200$. Then we have

$$\left(\frac{4}{5}\right)^{\frac{t}{90}} = \frac{2}{5}$$

$$\frac{t}{90} = \log_{\frac{4}{5}} \frac{2}{5}$$

$$t = 90 \log_{\frac{4}{5}} \frac{2}{5}.$$

(f) We have

$$90 \log_{\frac{4}{5}} \frac{2}{5} = 90 \cdot \frac{\ln \frac{2}{5}}{\ln \frac{4}{5}} = 90 \cdot \frac{\ln 2 - \ln 5}{2 \ln 2 - \ln 5}$$
$$\approx 90 \cdot \frac{0.69 - 1.61}{1.38 - 1.61} = 90 \cdot \frac{-0.92}{-0.23} = 90 \cdot 4 = 360$$

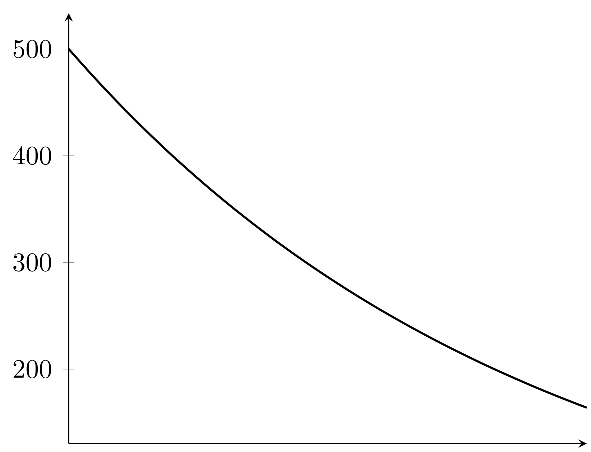
(g) We need to solve the equation $\left(\frac{4}{5}\right)^{\frac{t}{90}} = \frac{1}{2}$. Then we have

$$\left(\frac{4}{5}\right)^{\frac{t}{90}} = \frac{1}{2}$$
$$\frac{t}{90} = \log_{\frac{4}{5}} \frac{1}{2}$$
$$t = 90 \log_{\frac{4}{5}} \frac{1}{2}.$$

(h) We have

$$90 \log_{\frac{4}{5}} \frac{1}{2} = 90 \cdot \frac{\ln \frac{1}{2}}{\ln \frac{4}{5}} = 90 \cdot \frac{-\ln 2}{2 \ln 2 - \ln 5}$$
$$\approx 90 \cdot \frac{-0.69}{1.38 - 1.61} = 90 \cdot \frac{-0.69}{-0.23} = 90 \cdot 3 = 270$$

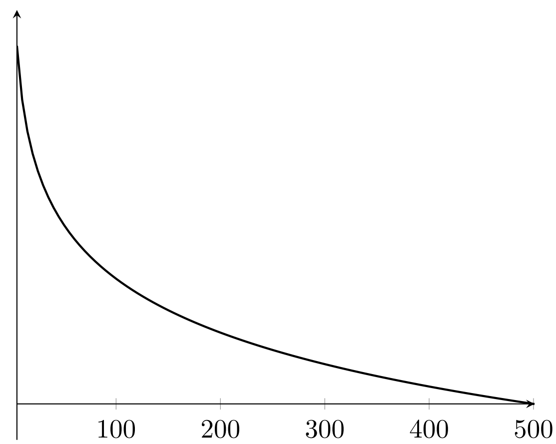
(i) $A(t)$ is invertible because the graph passes the horizontal line test.



(j) First of all we have $x = A(t) = 500 \cdot \left(\frac{4}{5}\right)^{\frac{t}{90}}$. To find the inverse function we need to solve for t.

$$\begin{aligned} x &= 500 \cdot \left(\frac{4}{5}\right)^{\frac{t}{90}} \\ \frac{x}{500} &= \left(\frac{4}{5}\right)^{\frac{t}{90}} \\ \log_{\frac{4}{5}} \frac{x}{500} &= \frac{t}{90} \\ t &= 90 \cdot \log_{\frac{4}{5}} \frac{x}{500} \end{aligned}$$

Hence $A^{-1}(x) = 90 \cdot \log_{\frac{4}{5}} \frac{x}{500}$.



$A^{-1}(x)$ describes the time at which there are x mg of acetaminophen in the body.

(k) We have

$$A^{-1}(x) = 90 \cdot \log_{\frac{4}{5}} \frac{x}{500} = 90 \cdot \left(\log_{\frac{4}{5}} x - \log_{\frac{4}{5}} 500\right).$$

Hence

$$(A^{-1})'(x) = 90 \cdot \frac{1}{\ln \frac{4}{5} \cdot x}.$$

(l) When there are 200 mg of acetaminophen in the body, it takes roughly $-(A^{-1})'(200)$ minutes to dissipate 1 mg of acetaminophen.