Solving Equations with Exponentials and Logarithms

Math Ma — Fall 2016

For each of the equation below, solve for x.

1. Warm up.
   (a) $7^x = 12$ (b) $x^7 = 12$ (c) $\log_7 x = 12$

2. These involve a single base.
   (a) $7^{4x+3} = 12$ (b) $5 \cdot 7^{x+3} = 7^{2x+1}$ (c) $7^x - \dfrac{49}{7^{x^2}} = 0$

3. These involve multiple bases. You do not need to simplify the answers as they are ugly.
   (a) $4^{x+3} = 5^{2x-1}$ (b) $7 \cdot 4^{x+2} = \dfrac{5}{3^{2x+1}}$ (c) $2^{x^2} = 5 \cdot 3^x$

4. These are easiest with a substitution.
   (a) $e^x(e^x - 5) = 6$ (b) $2e^{2x} + 6 = 7e^x$ (c) $7^x - \dfrac{8}{7^x} = 7$

5. More practice.

(a) $2^{2x+3} = 8^{x-7}$

(b) $3^x \cdot \frac{5}{3^{x+1}} = 0$

(c) $C^x = 3^{x+7}$

(d) $B^{2x} = 2^{Bx}$

(e) $3^{2x-1} - 3^x = \frac{10}{3}$

(f) $4^{\log_2 x} = 7$

Solving Equations with Exponentials and Logarithms – Solutions

1. (a) $x = \log_7 12$
   (b) $x = \sqrt[7]{12}$
   (c) $x = 7^{12}$

2. (a) Taking $\log_7$ for both sides, we have $4x + 3 = \log_7 12$. Hence

$$x = \frac{1}{4}(\log_7 12 - 3)$$

   (b) Putting the base 7 terms on the same side, we have $5 = 7^{x-2}$. Hence

$$x = \log_7 5 + 2$$

   (c) After expressing everything in base 7, we have $7^x = 7^{2-x^2}$, so the exponents must be equal: $x = 2 - x^2$. Solving $0 = x^2 + x - 2 = (x+2)(x-1)$, we have $x = -2, 1$.

3. (a) Take ln for both sides. Then $(x+3) \cdot \ln 4 = (2x-1) \cdot \ln 5$. Putting the terms involving x on one side, we have $(\ln 4 - 2\ln 5)x = -3\ln 4 - \ln 5$. Hence

$$x = \frac{-3\ln 4 - \ln 5}{\ln 4 - 2\ln 5}$$

   (b) We do the same thing as the previous problem.

$$\begin{aligned}\ln 7 + (x+2) \cdot \ln 4 &= \ln 5 - (2x+1) \cdot \ln 3 \\ (\ln 4 + 2\ln 3)x &= \ln 5 - \ln 3 - \ln 7 - 2\ln 4 = \ln 5 - \ln 336 \\ x &= \frac{\ln 5 - \ln 336}{\ln 4 + 2\ln 3}\end{aligned}$$

   (c) Taking $\log_2$ for both sides, we have $x^2 = \log_2 5 + x \cdot \log_2 3$. Hence we have the quadratic equation

$$x^2 - \log_2 3 \cdot x - \log_2 5 = 0$$

   The quadratic formula gives

$$x = \frac{\log_2 3 \pm \sqrt{(\log_2 3)^2 + 4\log_2 5}}{2}$$

4. (a) We let $y = e^x$. The equation becomes $y(y-5) = 6$, or $y^2 - 5x - 6 = 0$. Hence we have $(y-6)(y+1) = 0$, namely $y = e^x = 6, -1$. As exponential function is always positive, we only have one solution $x = \ln 6$.
   (b) Again we let $y = e^x$, and the equation becomes $2y^2 - 7y + 6 = 0$, or $(2y-3)(y-2) = 0$. Hence $y = e^x = \frac{3}{2}, 2$. We have $x = \ln \frac{3}{2} or \ln 2$.
   (c) We let $y = 7^x$, and the equation becomes $y^2 - 7x - 8 = 0$, or $(y-8)(y+1) = 0$. Hence $y = 7^x = 8, -1$, or $x = \log_7 8$.

5. (a) Writing everything in base 2, we have $2^{2x+3} = 2^{3x-21}$. Equating the exponent gives us $2x + 3 = 3x - 21$, or $x = 24$.

   (b) The equation simplifies to $\frac{5}{3} = 0$, which is impossible.

   (c) Taking ln for both sides, we have

$$\begin{aligned} x \cdot \ln C &= (x + 7) \cdot \ln 3 \\ x \cdot (\ln C - \ln 3) &= 7 \ln 3 \\ x &= \frac{7 \ln 3}{\ln C - \ln 3} \end{aligned}$$

   (d) This equation is always true if $B^2 = 2^B$, and always false if $B^2 \neq 2^B$. Hence in the former case any number is a solution, and in the latter case there is no solution.

   (e) Let $y = 3^x$. Then the equation is $\frac{y^2}{3} - y = frac103$, or $0 = y^2 - 3y - 10 = (y - 5)(y + 2)$. Hence $y = 3^x = 5, -2$, or $x = \log_3 5$.

   (f) The left hand side simplifies to x^2, so the equation is $x^2 = 7$, or $x = \pm\sqrt{7}$. However, looking at the original equation, since the domain of $\log_2$ is $(0, \infty)$, x can only be positive. Hence we only have one solution $x = \sqrt{7}$.