More Optimization

1. You need to submit the correction of the Exponential Gateway Exam for your Math Ma class. (Do it if you have not!) Chi-Yun insists that your correction has to be written on pieces of papers which are 200 square inches. They should have 1 inch margins on the sides, a 3 inch margin on the top for writing names and a 1 inch margin on the bottom. Say you want to use as few papers as possible. What dimensions of paper will give the largest writing area on one piece of paper?

2. A plane is chartered for a flight to New Zealand. The flight is booked for 100 passengers at $1000 per passenger. The price per ticket will be reduced by $5 per passenger for each person in excess of 100 who goes on the trip. Find the number of passengers that will provide maximum revenue. The plane will accommodate up to 140 passengers. (Revenue is the total amount of money made by the airline based on the number of passengers and the price per passenger.)

Optimization and Exponential Functions

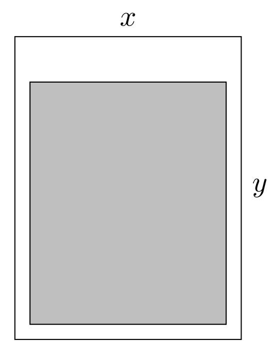
3. Chandler is looking at the graph $y = e^{-2x}$ and doodling rectangles. Each rectangle that she draws has its lower-left vertex at the origin and its upper-right vertex on the curve $y = e^{-2x}$. What is the largest possible area of such a rectangle?

4. Let $f(x) = x^3 e^{-x}$. Find the zeros, local extrema, and inflection points of f, as well as its asymptotic behavior as $x \to \pm\infty$. Then use what you find to sketch a graph of f.

5. A company is manufacturing smartphones at a cost of \$100 per phone. Their market research suggests that, if they price the phones at \$200 per phone, then they will sell 10,000 phones a month. For each \$5 increase in price, the number of phones they will sell decreases by 3%. The company wants to maximize revenue. Help them write down the revenue function.

More Optimization – Solutions

1. (a) Understand the problem: Below is the picture of the paper. The gray portion is the writing area.



   (b) State the goal: We want to minimize the gray area

$$A = (x-2)(y-4)$$

   where $x \geq 2$ and $y \geq 4$.

   (c) Do math: We have to make the area $A = (x-2)(y-4)$ single variable first. We know the area of the paper is $xy = 200$, so $y = \frac{200}{x}$. Hence we need to maximize

$$A(x) = (x-2)(\frac{200}{x} - 4) = 208 - \frac{400}{x} - 4x,$$

   where $x \geq 2$ and $y \geq 4$ implies $x \leq 50$. The derivative $A'(x) = -4 + \frac{400}{x}^2$, which is zero when $x = \pm 10$. We have the sign chart

x	2	10	50
$A'(x)$	$-$	0	$+$

   So $x = 10$ gives the global maximum. In this case $y = \frac{200}{x} = 20$.

2. Let x be the number of additional passengers above 100. Then the revenue function is

$$R(x) = (1000 - 5x)(100 + x) = -5x^2 + 500x + 100000$$

   where $0 \leq x \leq 40$. The derivative is

$$R'(x) = -10x + 500 = -10(x - 50).$$

   The zero of $R'(x)$ is $x = 50$, which is outside the domain $[0, 40]$. Hence the critical points of $R(x)$ are only the endpoints $x = 0$ and $x = 40$. Now we have $R(0) = 100000$ and $R(40) = (1000 - 5 \cdot 40) \cdot 140 = 112000$, so the maximum revenue happens when there are 140 passengers, and the revenue is \$1120000.

3. The area of the rectangle is $A(x) = xe^{-2x}$, where x should be greater than zero. We compute the derivative $A'(x) = e^{-2x} - 2xe^{-x} = e^{-2x}(1-2x)$, so the only critical point is $x = \frac{1}{2}$. Looking at the sign chart of $f'(x)$, we see this is a local maximum and global maximum. So the largest possible area is $\frac{1}{2} \cdot e^{-1} = \frac{1}{2e}$.

4. - There is only one zero of f, namely $x = 0$.
   - To find the local extrema, we first compute

$$f'(x) = 3x^2e^{-x} - x^3e^{-x} = (3x^2 - x^3)e^{-x} = -x^2(x-3)e^{-x}.$$

Drawing the sign chart

x		0		3	
$f'(x)$	+	0	+	0	−

we see that $x = 3$ is a local maximum and there is no local minimum.

   - An inflection point is where $f(x)$ changes concavity, or when $f''(x)$ changes sign. In particular, the second derivative f'' is zero at an inflection point. (Note that second derivative being zero does not always imply it is an inflection point.) We compute the second derivative

$$f''(x) = (6x - 3x^2)e^{-x} - (3x^2 - x^3)e^{-x} = (x^3 - 6x^2 + 6x)e^{-x} = x(x^2 - 6x + 6)e^{-x},$$

which is zero when $x = 0, 3-\sqrt{3}, 3+\sqrt{3}$. Again we draw the sign chart, but this time for $f''(x)$

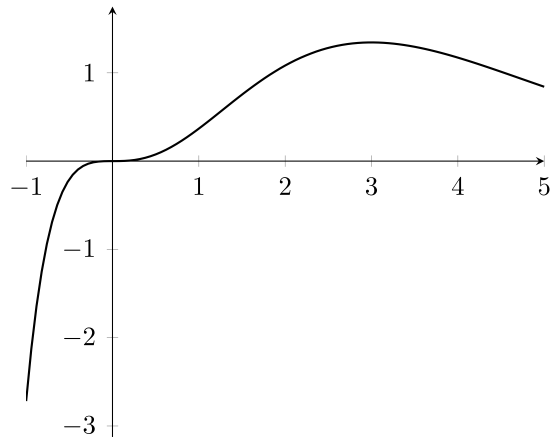
x		0		$3-\sqrt{3}$		$3+\sqrt{3}$	
$f''(x)$	−	0	+	0	−	0	+

Hence all three $x = 0, 3-\sqrt{3}, 3+\sqrt{3}$ are inflection points.

   - For the asympotic behavior of f, we have

$$\lim_{x\to\infty} x^3e^{-x} = 0, \quad \lim_{x\to-\infty} x^3e^{-x} = -\infty$$

Hence we may sketch the graph of $f(x) = x^3e^{-x}$ as below



5. Let the phones be priced at $200 + 5x$ dollars. Then they will sell $10000 \cdot 0.97^x$ phones. Hence the revenue function will be

$$R(x) = (200 + 5x) \cdot 100000 \cdot 0.97^x.$$