Math Ma

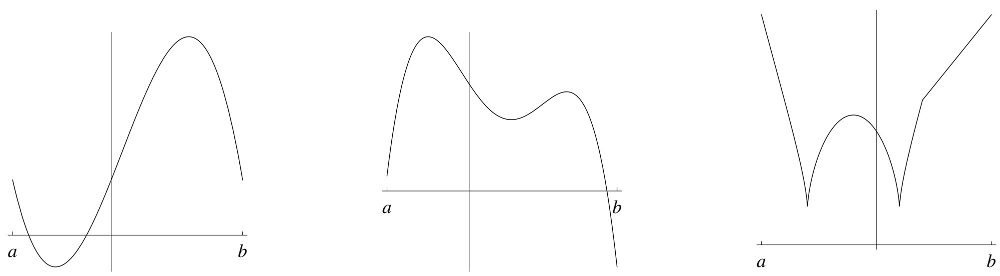
More on Global Extrema

Fall 2016

Critical Points. The critical points of f are all a in the domain of f such that

- $f'(a) = 0$,
- $f'(a)$ does not exist, or
- a is an endpoint of the domain.

Extremal Value Theorem. A continuous function on a closed interval must attain a global maximum and global minimum in the interval.



Strategy for finding global extrema on $[a, b]$:

Strategy for finding global extrema on (a, b):

Strategy for finding global extrema on $(-\infty, \infty)$:

1. Let $f(x) = \frac{x^2}{e^x}$ be a function defined on $[-2, 3]$.

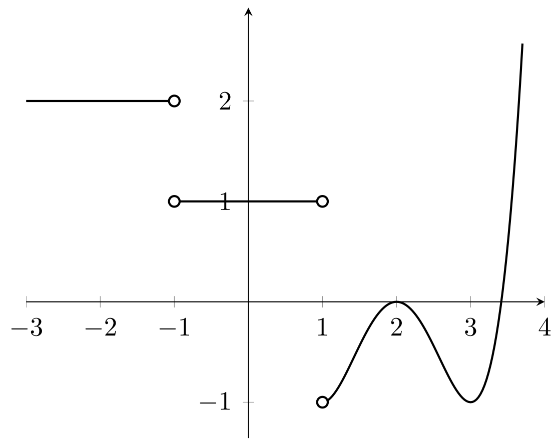
   (a) Find all critical points of f.

   (b) Find all local extrema of f.

   (c) Find the global extrema of f if they exist.

   (d) Now we consider f as a function on $(-\infty, \infty)$, find the global extrema of f if they exist.

2. Below is the graph of f'.



(a) Where are the critical points of f?

(b) Identify the local extrema of f.

(c) Identify the inflection points of f.

3. Let $f(r) = 2\pi r^2 + \frac{256\pi}{r}$.

(a) Find the global extrema of f on $[1, 8]$.

(b) Find the global extrema of f on $(0, \infty)$ if they exist.

4. An aluminum soft drink can has a volume of 128π cubic centimeters. In order to conserve resources, a soda company wants to minimize the amount of aluminum needed for a single can. What dimension should they make their cans?

More on Global Extrema – Solutions

1. (a) We first compute $f'(x) = 2xe^{-x} - x^2e^{-x} = -x(x-2)e^{-x}$. The first type of critical points are those when $f'(x)$ is zero, so we have $x = 0, 2$. The second type of critical points are those when $f'(x)$ does not exist. We have none of those. The third type of critical points are end points. We have $x = -2, 3$. Hence in total we have four critical points $x = -2, 0, 2, 3$.

   (b) We draw the sign chart of $f'(x)$.

x	-2		0		2		3
$f'(x)$		$-$	0	$+$	0	$-$	

   Hence $x = 0, 3$ are local minimum and $x = -2, 2$ are local maximum.

   (c) We are finding global extrema on a closed interval, so the Extremal Value Theorem gaurantees the existence on global extrema. Because $f(0) = 0, f(3) = \frac{9}{e^3} > 0$, the global minimum is at $x = 0$. On the other hand $f(-2) = 4e^2 > f(2) = 4e^{-2}$, so the global maximum is at $x = -2$.

   (d) In this case we only have two critical points $x = 0, 2$, which are local minimum and local maximum respectively. We need to look at the asymptotic behavior to make sure whether the global extrema exist. We have

   $$\lim_{x\to\infty} f(x) = 0 \qquad \lim_{x\to-\infty} f(x) = \infty.$$

   Hence the global minimum exists at $x = 0$ and there is no global maximum.

2. (a) As we do not have endpoints here, the critical points of f are either when $f'(x)$ is zero or not defined. At $x = 2$ and $x = 3.3$ we have $f'(x) = 0$. At $x = -1$ and $x = 1$, we have $f'(x)$ is not defined. Hence the critical poitns are $x = -1, 1, 2, 3.3$.

   (b) The local minima of f correspond to points of f' which go from negative to positive, so $x = 3.3$ is this kind. The local maxima of f correspond to points of f' which go from positive to negative, so $x = 1$ is a local maximum.

   (c) The inflection points of f correspond to turning points of f', so $x = 2$ and $x = 3$ are inflection points.

3. The derivative of f is

   $$f'(r) = 4\pi r - \frac{256\pi}{r^2} = \frac{4\pi}{r^2}(r^3 - 64) = \frac{4\pi}{r^2}(r-4)(r^2 + 4r + 16).$$

   The zeros of f' is $r = 4$.

   (a) The critical points are $r = 1, 4, 8$. Looking at the sign chart of f' we know $r = 4$ is a local minimum and $r = 1, 8$ are local maxima. Hence $r = 4$ is the global minimum. As $f(1) = 2\pi + 25\pi = 258\pi, f(8) = 128\pi + 32\pi = 160\pi$, $r = 1$ is the global maximum.

(b) We compute the asymptotic behaviro of f.

$$\lim_{r \to 0} f(r) = \infty \quad \lim_{r \to \infty} = \infty$$

Hence $f(r)$ does not have global maximum, but has global minimum. The global minimum is the smallest local minimum, which is $r = 1$.

4. We first write down the function describing the surface area of the can in terms of the radius r. Suppose the height of the can is h. As the volume is fixed to be 128π, we have the equality $r^2\pi h = 128\pi$, so $h = \frac{128\pi}{r^2}$. The surface area of the can can be computed as $2 \cdot \pi r^2 + 2\pi r \cdot h = 2\pi r^2 + \frac{256\pi}{r}$. This is the function which we found global extrema in problem 3. So the global minimum on $(0, \infty)$ would be at $r = 4$, $h = \frac{128\pi}{r^2} = 8\pi$.