Math Ma

Analysis of Extrema

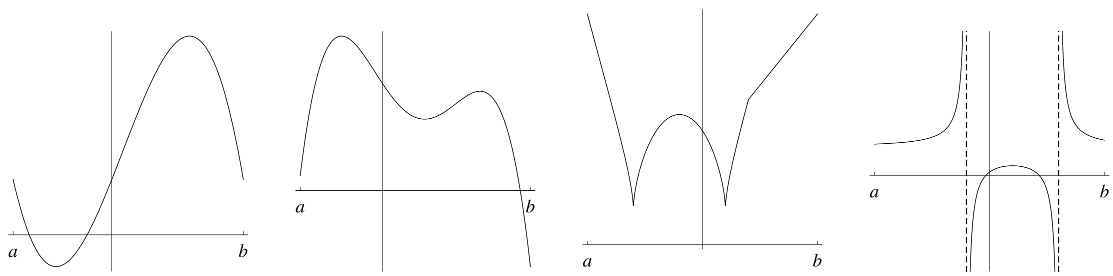
Fall 2016

1.

Local Extrema

- $f(x)$ has a local/relative maximum at $x = a$ if $f(a) \geq f(x)$ for all x near a.
- $f(x)$ has a local/relative minimum at $x = a$ if $f(a) \leq f(x)$ for all x near a.

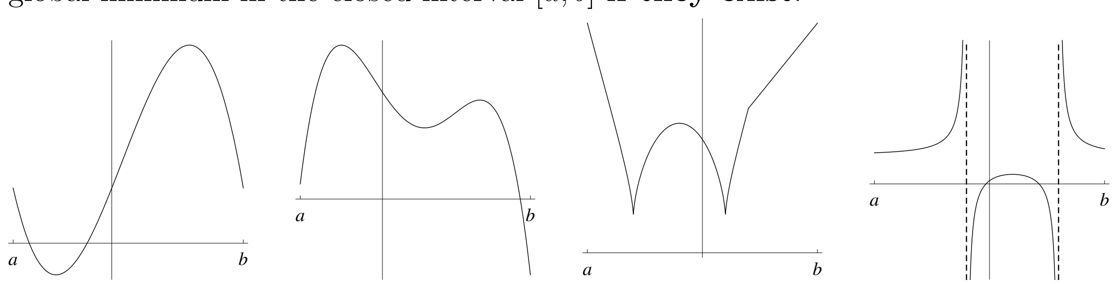
(a) For each of the following graphs, mark the local maxima and local minima in the closed interval $[a, b]$.



Global Extrema

- $f(x)$ has a global/absolute maximum at $x = a$ if $f(a) \geq f(x)$ for all x in the domain of f.
- $f(x)$ has a global/absolute minimum at $x = a$ if $f(a) \leq f(x)$ for all x in the domain of f.

(b) For the same set of graphs, reproduced below, mark the global maximum and global minimum in the closed interval $[a, b]$ **if they exist**.



(c) Now mark the global maximum and minimum for the above graphs in the open interval (a, b).

Critical Points. The critical points of f are all a in the domain of f such that

- $f'(a) = 0$,
- $f'(a)$ does not exist, or
- a is an endpoint of the domain.

Extremal Value Theorem. A continuous function on a closed interval must attain a global maximum and global minimum in the interval.

2. Let $f(x) = x^4 - 8x^2 + 16$.

   (a) Find all critical points of f.

   (b) Using only the first derivative f', determine whether each of the critical points is a local maximum, a local minimum, or neither.

   (c) Alternatively, use the second derivative f'' to classify the critical points. (Your answers to (b) and (c) should agree. We thus find all local extrema of f.)

   (d) Find the global extrema of f if they exist.

3. Let $f(x) = x^3$, $g(x) = x^4$.

   (a) Sketch the graph of f and g.

   (b) What are the critical points of f and g? Can you use the second derivative test to say whether the critical points are local maximum, local minimum, or neither?

4. Let $f(x) = 12x^5 - 15x^4 - 40x^3 + 7$ be a function defined on $[-2, 1]$.

   (a) Find all critical points of f.

   (b) Find all local extrema of f.

   (c) Find the global extrema of f if they exist.

5. Find the global extrema of $f(x) = x - 2 + \frac{1}{x}$ on $(0, \infty)$ if they exist.

Analysis of Extrema – Solutions

1.

2. (a) Setting $f'(x) = 4x^3 - 16x = 4x(x+2)(x-2)$ equal to zero, we have three critical points $x = -2, 0, 2$. As there are no points which f' does not exist, nor are there endpoints, these three points are the only critical points.

   (b) We may write down the sign chart of the derivative

x		-2		0		2	
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$

   This tells us that $f(x)$ is decreasing when $x < -2$ and increasing when $x > -2$, so $x = -2$ is a local minimum. Similarly, $x = 0$ is a local maximum, and $x = 2$ is a local minimum.

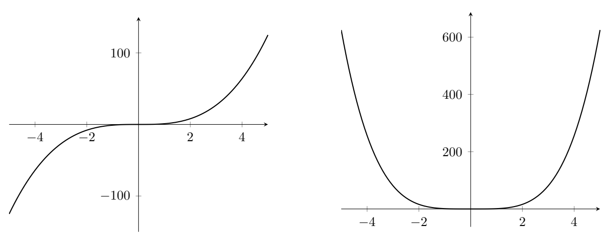
   (c) We have $f''(x) = 12x^2 - 16$. At the critical points, the second derivative is $f''(-2) = 32 > 0$, $f''(0) = -16 < 0$ and $f'(2) = 32 > 0$. Hence $f(x)$ is concave up at $x = -2$, meaning it is a local minimum. Similarly $x = 0$ is a local maximum and $x = 2$ is a local minimum.

   (d) f has domain $(-\infty, \infty)$, so we have to do extra work to ensure whether the global extrema exist. We have

$$\lim_{x \to \infty} f(x) = \infty \qquad \lim_{x \to -\infty} f(x) = \infty,$$

   so there cannot be global maximum and there is a global minimum. To find the global minimum, we need to compare the local minima and find the smallest one. We have $f(-2) = f(2) = 0$ is the local minimum.

3. (a)



   (b) $f'(x) = 3x^2$,$g'(x) = 4x^3$, so the critical points of f and g are only $x = 0$. Now $f''(x) = 6x$, $g''(x) = 12x^2$ which are both 0 at the critical point $x = 0$, so we cannot use the second derivative test to classify the critical points. Notice that the second derivative being zero at a critical point does not mean the critical point is not a local extrema. It simply means we cannot use the second derivative test to conclude.

4. (a) $f'(x) = 60x^4 - 60x^3 - 120x^2 = 60x^2(x-2)(x+1)$. The zeros of $f'(x)$ in the domain $[-2, 1]$ are $x = -1, 0$. There are no point which $f'(x)$ is not defined. There are end points $x = -2, 1$ that we should also consider. Hence the critical points are $x = -2, -1, 0, 1$.

(b) Using the sign chart of f'

x	-2		-1		0		1
$f'(0)$		$+$	0	$-$	0	$-$	

we may conclude that $x = -2$ is a local minimum, $x = -1$ is a local maximum, $x = 0$ is not a local extrema and $x = 1$ is a local minimum.

(c) Our f is defined on a closed interval, so the Extremal Value Theorem tells us that the global extrema exist. Then the global maximum is simply the largest local maxima and the global minimum is the smallest local minimum. So the global maximum is at $x = -1$. Since $f(1) = -36 > f(-2) = -297$, the global minimum is at $x = -2$.

5. The domain is an open interval, so we need to look at the asymptotic behavior to determine whether the global extrema exist. We have

$$\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = \infty,$$

so the global maximum does not exist and the global minimum exist. The derivative $f'(x) = 1 - \frac{1}{x^2}$. So there is only one critical point in the domain $(0, \infty)$, namely $x = 1$. This is the global minimum of f.