Cubics and Higher-Degree Polynomials

1. For each description below,

   - sketch the graph of a cubic function $f(x)$ with the specified characteristics.
   - find a formula for $f(x)$

   There may be many possible answers, or none at all.

   (a) $f(x)$ has zeros at $x = -\pi$, $x = 2$, and $x = 5$.

   (b) $f(x)$ has only two zeros, at $x = 2$ and $x = 5$, and a y-intercept of -4.

   (c) $f(x)$ has only one zero, at $x = 2$, and passes through the point $(1, 5)$.

   (d) $f(x)$ has no zero, and passes through the points $(2, 3)$, $(5, -1)$.

2. Sketch the graph of $y = x(x + 2)(x - 2)$. Where are the points having horizontal tangent lines?

3. Find a cubic polynomial with local minimum at $x = 2$ and local maximum at $x = 5$. Where is the inflection point?

4. For each description below,

   - sketch the graph of a polynomial function $P(x)$ with the specified characteristics.
   - find a formula for $P(x)$

   There may be many possible answers, or none at all.

   (a) A degree 4 polynomial with a zero of multiplicity two at $x = 1$, zeros at $x = 5$ and $x = e$, and a y-intercept of 7.

   (b) A degree 5 polynomial with no zeros and $\lim_{x \to \infty} P(x) = -\infty$.

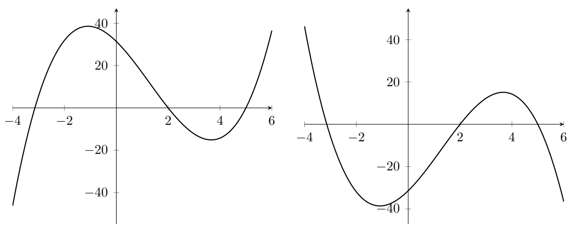
   (c) A degree 6 polynomial with no zeros and $\lim_{x \to \infty} P(x) = -\infty$.

5. (a) How many zeros can a degree n polynomial have at most? At least?

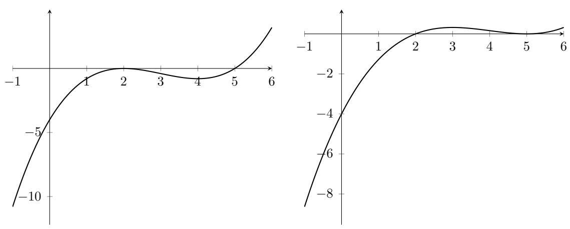
   (b) How many points with horizontal tangent line can a degree n polynomial have at most? At least?

Cubics and Higher-Degree Polynomials – Solutions

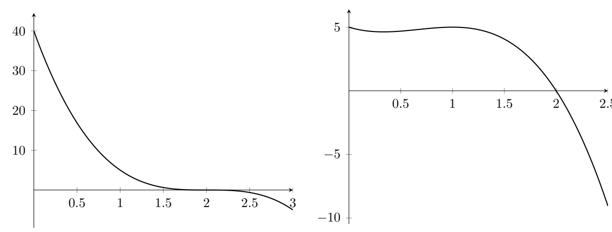
1. (a) $f(x) = a(x+\pi)(x-2)(x-5)$



(b) $f(x) = a(x-2)^2(x-5)$ or $f(x) = b(x-2)(x-5)^2$. We have $f(0) = -4$, so $-4 = a \cdot 4 \cdot (-5)$ and $-4 = b \cdot (-2) \cdot 25$. Hence $a = \frac{1}{5}$ and $b = \frac{2}{25}$.

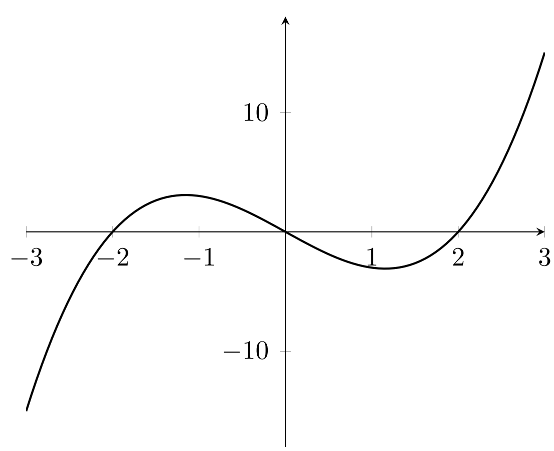


(c) $f(x) = a(x-2)^3$ or $f(x) = a(x-2)Q(x)$ where $Q(x)$ is a quadratic polynomial without zero.



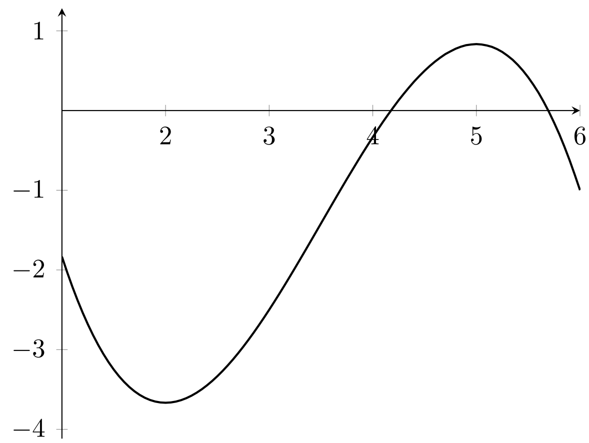
(d) By intermediate value theorem, there is no cubic polynomial with no zero.

2.

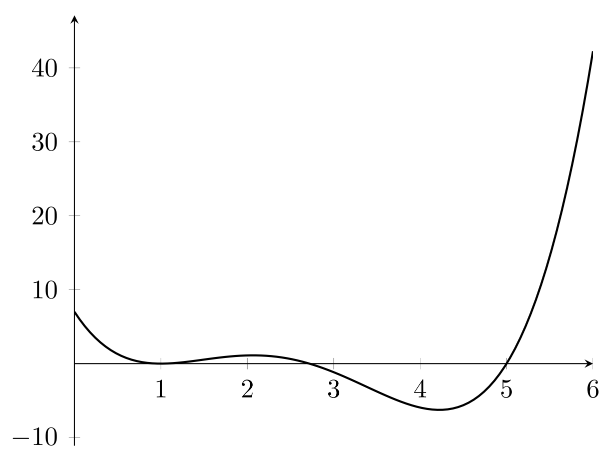


We have cubic function $f(x) = x(x+2)(x-2) = x^3 - 4x$. The derivative is $f'(x) = 3x^2 - 4$, which is zero when $x = \pm\frac{2}{\sqrt{3}}$. Hence the points with horizontal tangent lines are $x = \pm\frac{2}{\sqrt{3}}$.

3. The cubic polynomial has local extremum at $x = 2$ and $x = 5$ means that the derivative has zeros at $x = 2, 5$. Hence $f'(x) = a(x-2)(x-5)$. We know a should be negative because $f(x)$ is increasing between $[2, 5]$. Then $f(x)$, as the antiderivative of $f'(x) = a(x^2 - 7x + 10)$, is $a(\frac{1}{3}x^3 - \frac{7}{2}x^2 + 10x)$, where a is negative.

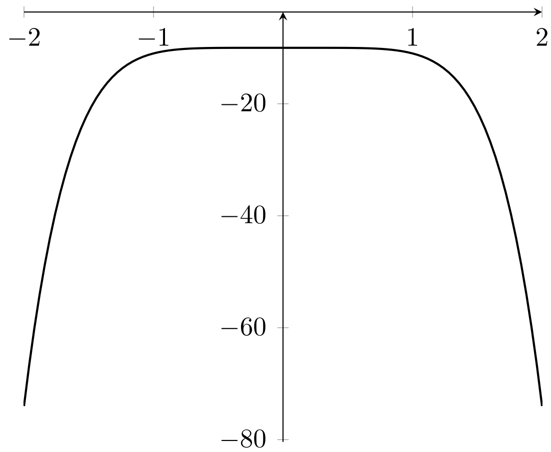


4. (a) $P(x) = a(x-1)^2(x-5)(x-e)$. Since $P(0) = 7$, we have $7 = a \cdot 5e$ and $a = \frac{7}{5e}$.



(b) A degree 5 polynomial must have a zero.

(c) We can have $P(x) = -x^6 - 10$.



5. (a) A degree n polynomial has at most n zeros. If n is odd, then it has at least one zero. If n is even, it may happen that it has no zeros at all.

   (b) Points with horizontal tangent lines correspond to zeros of the derivative, which is a degree $n-1$ polynomial. Hence there can be at most $n-1$ points with horizontal tangent lines. If n is even, there is at least one such points. If n is odd, there can be no such point at all.