Math Ma

Continuity

Fall 2016

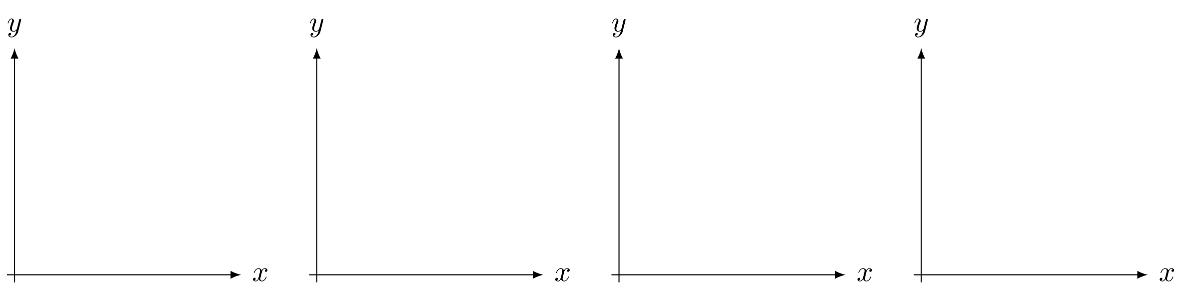
1. Sketch a graph of a discontinuous function. How would you describe a continuous function?

Definition

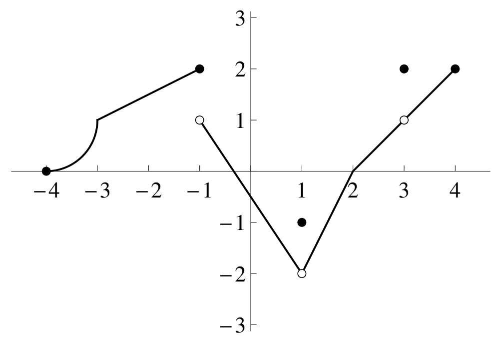
A function $f(x)$ is continuous at $x = a$ if

Different Types of discontinuity

- Removable: ____________________
- Jump: ____________________
- Vertical Asymptote: ____________________
- Oscillation: ____________________



2. The graph of a function $g(x)$ is given below. Evaluate the following limits.



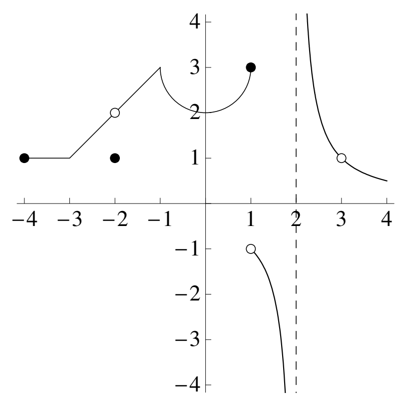
(a) $\lim_{h \to 0} \dfrac{g(-2+h) - g(-2)}{h}$

(b) $\lim_{x \to 0} \dfrac{g(x) - g(0)}{x}$

(c) $\lim_{x \to 2} \dfrac{g(x) - g(2)}{x - 2}$

(d) $\lim_{x \to 3} \dfrac{g(x) - g(3)}{x - 3}$

3. Here is the graph of a function $f(x)$.



(a) At what values of x in $(-4, 4)$ is $f(x)$ not continuous? Specify the type of discontinuities.

(b) At what values of x in $(-4, 4)$ is $f(x)$ not differentiable?

(c) Which of the following statements are true?

- If $f(x)$ is continuous at $x = a$, then $f(x)$ is differentiable at $x = a$.
- If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

4. Which of the following functions are continuous at $x = 1$? Which are differentiable at $x = 1$?

(a) $f(x) = \begin{cases} x^2 & x \geq 1 \\ 0 & x < 1 \end{cases}$

(b) $f(x) = \begin{cases} x^2 & x \geq 1 \\ 1 & x < 1 \end{cases}$

(c) $f(x) = \begin{cases} x^2 & x \geq 1 \\ x & x < 1 \end{cases}$

(d) $f(x) = \begin{cases} x^2 & x \geq 1 \\ 2x - 1 & x < 1 \end{cases}$

5. Let f be the function defined by

$$f(x) = \begin{cases} -x^2 + 1 & \text{for } x \leq 1 \\ ax + b & \text{for } x > 1 \end{cases}$$

(a) For what values of a and b will f be continuous at $x = 1$?

(b) For what values of a and b will f be differentiable at $x = 1$?

6. In each part, sketch, if possible, a **continuous** function $f(x)$ defined on $[-4, 3]$ which has all three of the given properties. If it is impossible, will it become possible if we do not require $f(x)$ to be continuous?

   (a)
   - $f(-4) = 2$
   - $f(3) = 5$
   - f has no zeros in $[-4, 3]$

   (b)
   - $f(-4) = 2$
   - $f(3) = 5$
   - f has a zero in $[-4, 3]$

   (c)
   - $f(-4) = 2$
   - $f(3) = 5$
   - f has exactly one zero in $[-4, 3]$

   (d)
   - $f(-4) = 2$
   - $f(3) = -1$
   - f has no zeros in $[-4, 3]$

   (e)
   - $f(-4) = 2$
   - $f(3) = -1$
   - f does not attain the value 1 on $[-4, 3]$

Intermediate Value Theorem

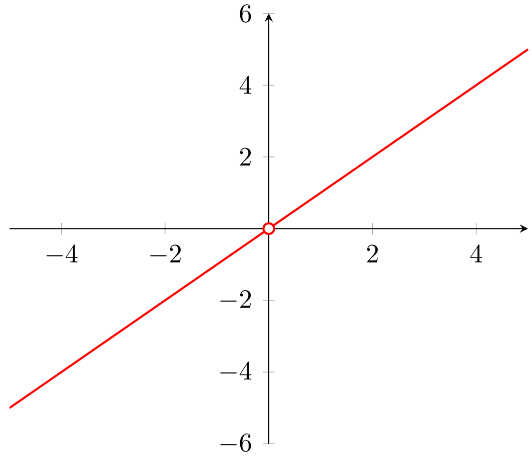
If f is continuous on the closed interval $[a, b]$ and $f(a) = A$, $f(b) = B$, then for any value C between A and B, there is some number c in $[a, b]$ such that $f(c) = C$.

7. Is the following argument correct or incorrect? Why?

   The Harvard football team beat Cornell 29-13 on Saturday. Since Harvard had 0 points at the beginning of the game and 29 at the end, the Intermediate Value Theorem says that Harvard must have had exactly 25 points at some moment in the game.

Continuity – Solutions

1. Here is an example of a discontinuous function:



2. (a) $\frac{1}{2}$
   (b) $-\frac{3}{2}$
   (c) Does not exist.
   (d) Does not exist.

3. (a) It has a removable discontinuity at $x = -2$, a jump discontinuity at $x = 1$, a vertical asymptote at $x = 2$, and another removable discontinuity at $x = 3$.
   (b) It is not differentiable at $x = -3$, $x = -2$, $x = -1$, $x = 1$, $x = 2$, and $x = 3$.
   (c) The first statement is false. For example, $f(x)$ is continuous at $x = -3$, but it is not differentiable at $x = -3$. ($x = -1$ is another counterexample.) Really, any place where $f(x)$ has a "corner" is a counterexample. The second statement is true.

4. For all four functions, $\lim_{x\to 1^+} f(x) = 1$ and $f(1) = 1$. So for a function to be continuous at $x = 1$ it must satisfy $\lim_{x\to 1^-} f(x) = 1$, where only (b)(c)(d) satisfy.

   Since (a) is not continuous at $x = 1$, it cannot be differentiable at $x = 1$. For others we compute $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = 2$. However, $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$ are 0, 1 and 2, respectively in (b), (c) and (d). Hence only (d) is differentiable.

5. (a) We have $\lim_{x\to 1^-} f(x) = 0$ and $\lim_{x\to 1^+} f(x) = a + b$. So whenever $a + b = 0$, the function is continuous at $x = 1$.
   (b) We have $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = -2$ and $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = a$. So when $a = -2$, $b = -a = 2$, the function is (continuous and) differentiable at $x = 1$.

6. Only (a)(b)(c) are possible.

7. The argument is incorrect. The score is not a continuous function of time, so the Intermediate Value Theorem does not apply.