Limits

1. Let $f(x) = x^2$.

   (a) Carefully write down each step of the calculation of $f'(1)$ from definition.

   (b) Sketch the graph of the function $\dfrac{f(x) - f(1)}{x - 1}$.

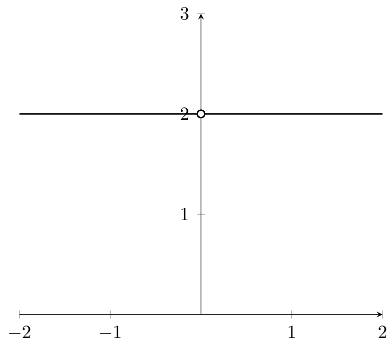
Definition

We write

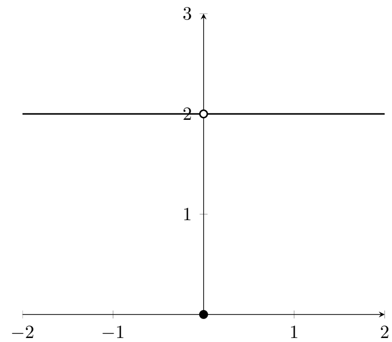
to mean "the number which $f(x)$ approaches as x gets really close, but not equal to, a". It is the limit of $f(x)$ as x approaches a.

2. What is $\lim_{x \to 0} f(x)$ in each of the following cases? How about $f(0)$?

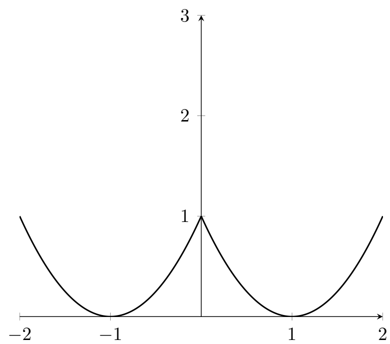
(a)



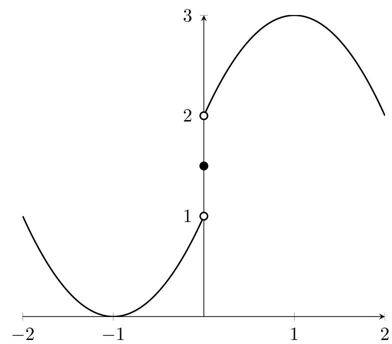
(b)



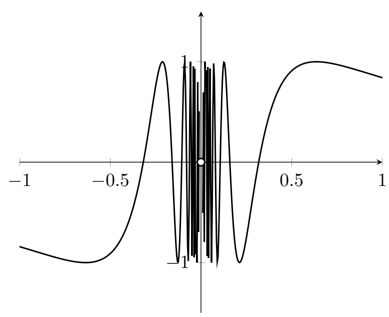
(c)



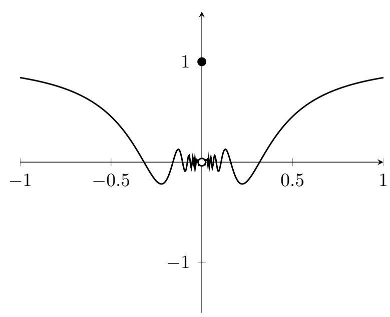
(d)



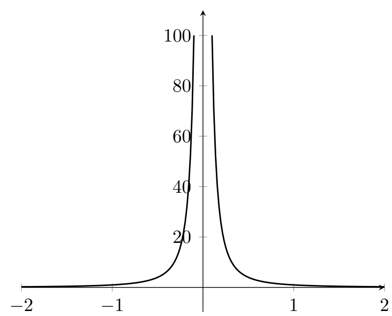
(e)



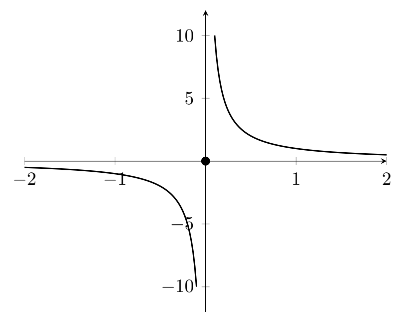
(f)



(g)



(h)



3. (Food for thought)

   Let

$$f(x) = \begin{cases} 1 & \text{if } x = \pm\frac{1}{2^n} \text{ for some integer } n \\ 0 & \text{otherwise} \end{cases}$$

   be a function defined for all real numbers. What is $\lim_{x\to 0} f(x)$?

Definition

- The <u>limit of $f(x)$ as x approaches infinity</u>

  [   ]

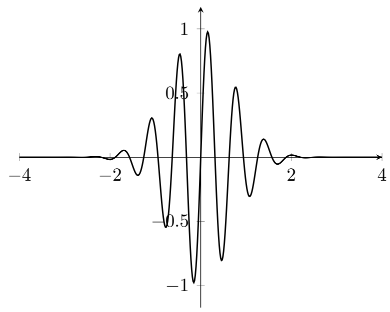
  is the number that $f(x)$ approaches as x gets arbitrarily large.
- The <u>limit of $f(x)$ as x approaches negative infinity</u>

  [   ]

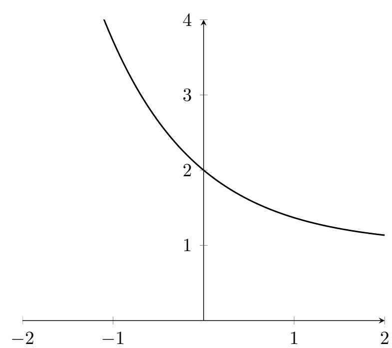
  is the number that $f(x)$ approaches as x gets arbitrarily negative.

4. What is $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ in each of the following cases?

   (a)



   (b)



Definition

- The limit of $f(x)$ as x approaches a from the left,

  is the number that $f(x)$ approaches as x gets really close to a, while remaining slightly less than a.

- The limit of $f(x)$ as x approaches a from the right,

  is the number that $f(x)$ approaches as x gets really close to a, while remaining slightly greater than a.

5. (a) Sketch the graph of $|x|$ and $\frac{|x|}{x}$.

Determine the followings

(b) $\lim_{x \to 0^+} \frac{|x|}{x}$ (c) $\lim_{x \to 0^-} \frac{|x|}{x}$ (d) $\lim_{x \to 0} \frac{|x|}{x}$

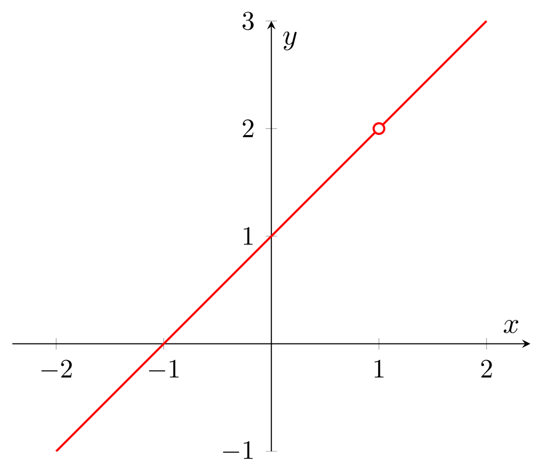
(e) What is the derivative of $f(x) = |x|$?

Limits – Solutions

1. (a)

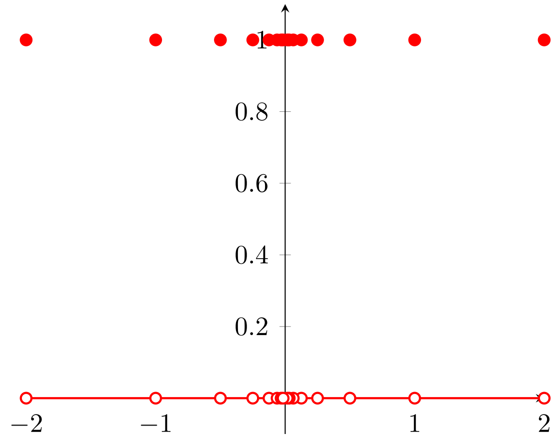
$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x - 1} = \lim_{x\to 1}(x+1) = 2$$

(b)



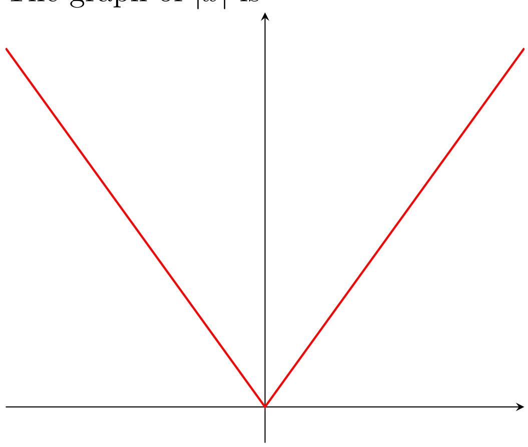
2. (a) $\lim_{x\to 0} f(x) = 2$, $f(0)$ does not exist.
   (b) $\lim_{x\to 0} f(x) = 2$, $f(0) = 0$.
   (c) $\lim_{x\to 0} f(x) = 1$, $f(0) = 1$.
   (d) $\lim_{x\to 0} f(x)$ does not exist, $f(0) = 1.5$.
   (e) $\lim_{x\to 0} f(x)$ does not exist, $f(0)$ does not exist.
   (f) $\lim_{x\to 0} f(x) = 0$, $f(0)$ does not exist.
   (g) $\lim_{x\to 0} f(x)$ does not exist (with the type of infinity), $f(0)$ does not exist.
   (h) $\lim_{x\to 0} f(x)$ does not exist, $f(0)$ does not exist.

3. $\lim_{x\to 0} f(x)$ does not exist. The reason is that no matter how close x gets to 0, if we get a little bit more closer to some $x = \frac{1}{2^n}$, $f(x)$ jumps away from 0 to 1. Here is an illustrating picture

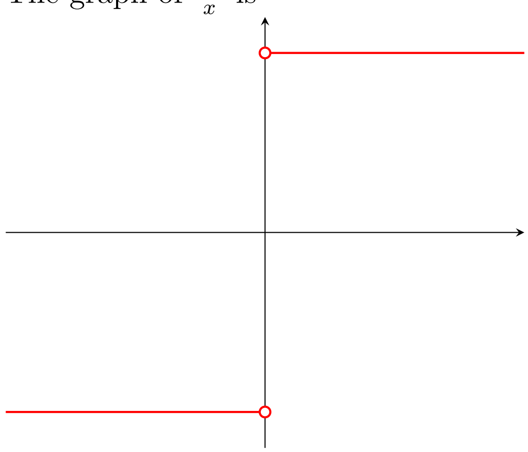


4. (a) $\lim_{x\to\infty} f(x) = 0, \lim_{x\to-\infty} f(x) = 0.$
   (b) $\lim_{x\to\infty} f(x) = 1, \lim_{x\to-\infty} f(x)$ does not exist.

5. (a) The graph of $|x|$ is



The graph of $\frac{|x|}{x}$ is



   (b) $\lim_{x\to 0^+} \frac{|x|}{x} = 1.$

(c) $\lim_{x\to 0^-} \frac{|x|}{x} = -1$.

(d) $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

(e)

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$