Algebra and Composition of Functions

If we have two functions f and g, how can we create new functions out of them?

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1. Geraldine is running a small chocolate store. She figures that the cost of making a box of chocolate in the t-th month is $c(t)$, which changes because of the price increase of the ingredients. In the t-th month, $N(t)$ boxes of chocolates were made and all sold out in her store. Let $R(t)$ be the function of total revenue in the t-th month.

   (a) What is the function $C(t)$ describing the total cost of making chocolate in the t-th month?

   (b) What is the function $P(t)$ describing the total profit in the t-th month?

   (c) What is the function $p(t)$ describing the average profit obtained from making and selling one box of chocolate in the t-th month?

   Now suppose you have a $5 coupon for the chocolate store and a membership card which gives you 30% off for any purchase.

   (d) What is the function $f(d)$ describing the amount you have to pay only applying the coupon, when the original price is d dollars?

   (e) What is the function $g(d)$ describing the amount you have to pay only applying the membership card, when the original price is d dollars?

   (f) If the original price is d, how much do you really have to play first applying the coupon and then the membership card? How about first applying the membership card and then the coupon? Do you pay the same using the two methods?

2. The following table gives values of two functions $f(x)$ and $g(x)$.

x	-2	-1	0	1	2	3	4
$f(x)$	0	4	1	3	4	-2	5
$g(x)$	6	3	4	1	3	-1	0

Use the table to find the following.

(a) $(f \cdot g)(3)$

(b) $(g \cdot f)(3)$

(c) $(f \circ g)(3)$

(d) $(g \circ f)(3)$

What do you think the above four problems want to tell you?

(e) All x so that $g(f(x)) = 0$

3. Let
$$f(x) = x^2, \quad g(x) = \frac{3}{x}, \quad h(x) = 2x - 1$$
Write out each of the following function. Also determine their domains.

(a) $(f + g)(x)$

(b) $(f - g)(x)$

(c) $(f \cdot g)(x)$

(d) $(f \div g)(x)$

(e) $(f \div h)(x)$

(f) $(g \div h)(x)$

(g) $(f \circ h)(x)$

(h) $(h \circ f)(x)$

(i) $(f \circ g \circ h)(x)$

(j) $(g \circ g)(x)$

Can you find a function $w(x)$ such that $g(w(x)) = x$? What is the domain of w?

4. A small candy store is planning to make a special batch of Halloween lollipops. They have determined that the total cost of making x lollipops is $C(x) = \sqrt{x}$ dollars, and the revenue (money they will take in) from making x lollipops is $R(x) = \frac{x}{4}$ dollars.

(a) *If the store decides to make a batch of 100 lollipops, what will their profit be?*

Solution. The cost of making 100 lollipops is $C(100) = 10$ dollars. The revenue from making 100 lollipops is $R(100) = 25$ dollars, so the profit is $25 - 10 = \boxed{15 \text{ dollars}}$.

(b) *What function describes the profit the company will get from making x lollipops assuming all lollipops made are sold?*

Solution. $R(x) - C(x)$, or $\boxed{(R - C)(x)}$ This is an example of how two functions we know combine in a useful way to create a new function.

(c) *What function describes the average profit per lollipop if x lollipops are made and all are sold?*

Solution. From the previous part we know that the profit is $(R - C)(x)$ if we sell x lollipops. To calculate the average profit per lollipop we divide by the total number of lollipops sold, x.

$$\boxed{\frac{(R - C)(x)}{x}}$$

What is the domain of this new function? Rob thinks that those x's cancel. Explain to Rob why he is wrong.

5. The price of tomatoes varies seasonally; suppose $P(x)$ gives the price of tomatoes in dollars per pound at time x. The demand for tomatoes (in other words, the amount of tomatoes people want to buy) depends on the price; say that $D(x)$ gives the demand for tomatoes when they cost $\$x$ per pound.

(a) *When tomatoes cost $\$3$ per pound, what is the demand for tomatoes?*

Solution. $\boxed{D(3)}$

(b) *At time 3, what is the demand for tomatoes?*

Solution. $\boxed{D(P(3))}$

(c) *What function gives the demand for tomatoes at time t?* **Solution.** $\boxed{D(P(t))}$

6. The fuel efficiency of a car (measured in miles per gallon) depends on the speed of the car; the faster the car is going, the lower its fuel efficiency. Suppose $f(x)$ gives the fuel efficiency of a car that is going x miles per hour.

Bob is driving from Boston to New York, a distance of 225 miles. The trip takes him 4 hours. Let $s(x)$ be Bob's speed (measured in miles per hour) x miles into the trip. Let $d(t)$ be the distance Bob has driven (measured in miles) t hours into the trip.

(a) *From the given information, what can you say about the domains and ranges of the functions $f(x)$, $s(x)$, and $d(t)$? (You do not have enough information to find all of them!)*

Solution. We can't say anything about the domain *or* range of $f(x)$: the domain is the set of possible speeds of the car, while the range is the set of possible fuel efficiencies. But we don't know either of those. You could make some reasonable guesses based on our experiences.

The domain of $s(x)$ is the set of possible "number of miles into the trip", which is $[0, 225]$. The range of $s(x)$ is the set of possible speeds, which we don't know.

The domain of $d(t)$ is the set of times in Bob's trip, or $[0, 4]$. The range of $d(t)$ is the set of possible distances, or $[0, 225]$.

(b) *Do you have enough information to tell where the functions $f(x)$, $s(x)$, and $d(t)$ are increasing and decreasing?*

Solution. The faster a car is going, the lower its fuel efficiency. So, $f(x)$ is a decreasing function (over its whole domain).

We can't tell when $s(x)$ is increasing or decreasing.

$d(t)$ is the distance Bob has driven t hours into the trip; this is always increasing.

(c) Write expressions for each of the following.

     i. *The fuel efficiency when Bob is going at 45 mph.*

     Solution. $\boxed{f(45)}$

     ii. *The fuel efficiency of Bob's car 1.5 hours into the trip.*

     Solution. Since we want fuel efficiency, we need to use the function f. But the input to f should be the speed of the car, so we ought to input the speed Bob is going after 1.5 hours. But the speed function s takes as input Bob's distance into the trip. So, we need to use the distance function to get that. Bob is 1.5 hours into the trip, so his distance is given by $d(1.5)$, his speed is

$s(d(1.5))$, and his fuel efficiency is $\boxed{f(s(d(1.5)))}$.

iii. *The fuel efficiency of Bob's car 123.7 miles into the trip.*

Solution. $\boxed{f(s(123.7))}$

iv. *Bob's speed 2.1 hours into the trip.*

Solution. $\boxed{s(d(2.1))}$

v. *The function giving the fuel efficiency of Bob's car t hours into the trip. (What can you say about the domain and range of this function?)*

Solution. $f(s(d(t)))$. The domain is the set of possible times, or $[0, 4]$. We don't know what the range is.

Algebra and Composition of Functions – Solutions

1. (a) The total cost is the cost of a box of chocolate times the number of boxes of chocolates made. So $C(t) = c(t) \cdot N(t)$.

   (b) The profit is the total revenue minus the total cost, so $P(t) = R(t) - C(t) = R(t) - c(t) \cdot N(t)$.

   (c) The average profit from obtained from one box of chocolate is $\frac{P(t)}{N(t)} = \frac{R(t)}{N(t)} - c(t)$.

   (d) $f(d) = d - 5$.

   (e) $g(d) = 0.7d$.

   (f) Applying the coupon first and then the membership card, you have to pay

   $$(g \circ f)(d) = 0.7(d - 5) = 0.7d - 3.5$$

   On the other hand, applying the membership card first, you pay

   $$(f \circ g)(d) = 0.7d - 5$$

   They are not the same and the second method is cheaper!

2. (a) 2

   (b) 2

   (c) 4

   (d) 6

   (e) Multiplication of functions is commutative, while composition of functions is not.

   (f) g outputs 0 exactly when the input is 4. And f outputs 4 exactly when the input is -1 or 2. Hence $x = -1, 2$.

3. (a) $(f + g)(x) = f(x) + g(x) = x^2 + \frac{3}{x}$. The domain is all real numbers except for 0.

   (b) $(f - g)(x) = f(x) - g(x) = x^2 - \frac{3}{x}$. The domain is all real numbers except for 0.

   (c) $(f \cdot g)(x) = f(x) \cdot g(x) = x^2 \cdot \frac{3}{x} = 3x$. The domain is all real numbers except for 0.

   (d) $(f \div g)(x) = f(x)/g(x) = x^2 \div \frac{3}{x} = \frac{x^3}{3}$. The domain is all real numbers except for 0.

   (e) $(f \div h)(x) = f(x)/h(x) = \frac{x^2}{2x-1}$. The domain is all real numbers except for $\frac{1}{2}$.

(f) $(g \div h)(x) = g(x)/h(x) = \frac{3}{x(2x-1)}$. The domain is all real numbers except for 0 and $\frac{1}{2}$.

(g) $(f \circ h)(x) = f(h(x)) = (2x - 1)^2$. The domain is all real numbers.

(h) $(h \circ f)(x) = h(f(x)) = 2x^2 - 1$. The domain is all real numbers.

(i) $(f \circ g \circ h)(x) = f(g(h(x))) = \left(\frac{3}{2x-1}\right)^2$. The domain is all real numbers except for $\frac{1}{2}$.

(j) $(g \circ g)(x) = g(g(x)) = \frac{3}{\frac{3}{x}} = x$. The domain is all real numbers except for 0.

$g(w(x)) = x$ means $\frac{3}{w(x)} = x$, so $w(x) = \frac{3}{x} = g(x)$. We actually have seen this in (j). The domain of w is all real numbers except for 0.