AUTOMORPHIC CUSPIDAL REPRESENTATIONS AND MAASS FORMS

(ENCOUNTERS AND RUMINATIONS)

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CHAT

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(THESE NOTES WERE UPDATED AFTER THE LECTURE TO CORRECT SOME POINTS).

OLDER HISTORY

H. MAASS (STUDENT OF SIEGEL)

EXTENDED SIEGEL'S WORK ON THETA FUNCTIONS:

$$G = SO_F(p,q),\ H = SL_2 \text{ OR } \widetilde{SL}_2 \,/\mathbb{Q}$$

$H \times G$ IS A DUAL PAIR

$\Theta(h,g)$ IS $\widetilde{\Gamma} \times G(\mathbb{Z})$ INVARIANT ON $H \times G$.

$$f(h) := \int_{G(\mathbb{Z})\backslash G(\mathbb{R})} 1 .\ \Theta(h,g)\, dg$$

WHEN SUMMED OVER THE GENUS OF F (I.E. ADELIC INTEGRAL) IS AN EISENSTEN SERIES ON H.

"SIEGEL-WEIL FORMULA"

- REPLACING 1 BY OTHER AUTOMORPHIC FORMS MAASS FINDS THAT $f(h)$ SATISFIES NATURAL DIFFERNTIAL EQUATIONS (AS DO EISENSTEIN SERIES!)

$\Longrightarrow$ MAASS FORMS.

SIMPLEST SETTING

$SL_2(\mathbb{R})$, Γ A DISCRETE SUBGROUP WITH FINITE VOLUME QUOTIENT.

$$\mathbb{H} = SL_2(\mathbb{R})/SO_2(\mathbb{R})$$ UPPER HALF PLANE.

$\Gamma\backslash\mathbb{H}$ NOT COMPACT.

SEEK SOLUTIONS TO

$$(*)\quad \begin{cases} \Delta\phi + \lambda\phi = 0, \quad \Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ \phi(\gamma z) = \phi(z) \\ \phi \in L^2(\Gamma\backslash\mathbb{H}) \end{cases}$$

MAASS PROVED THAT THE SPACE IS FINITE DIMENSIONAL, BUT COULD NOT PRODUCE ANY SUCH FORMS IN ANY GENERALITY.

BASIC EXAMPLE: $\Gamma = PSL_2(\mathbb{Z})$, $(PGL_2(\mathbb{Z}))$.

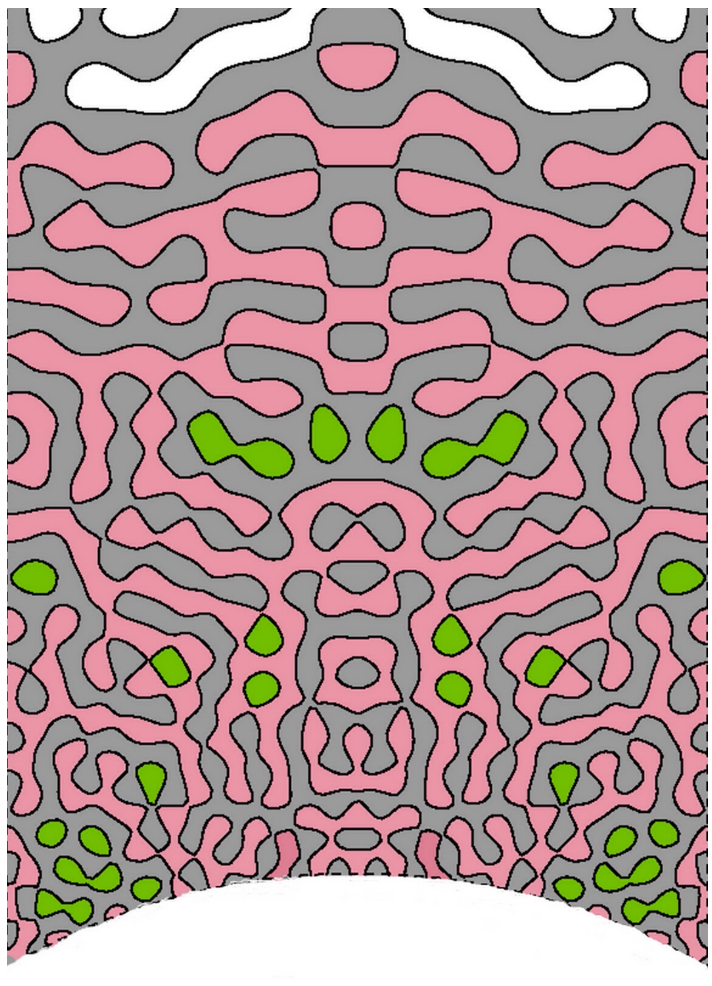
$\phi(-\bar{z}) = \phi(z)$ EVEN;

EVERYWHERE UNRAMIFIED CUSP FORMS FOR $GL_2/\mathbb{Q}$

CUSPIDAL CONDITION: $\int_0^1 \phi(x,y)\,dx = 0$ FOR ALL $y > 0$

(INVARIANT SUBSPACE L^2_{cusp}; SPECTRUM IS DISCRETE!)

Nodal portrait



<u>SELBERG</u>: DEVELOPED THE TRACE FORMULA AND THE ANALYTIC CONTINUATION OF EISENSTEIN SERIES, FOR THIS EXAMPLE IN ORDER TO PROVE THAT SUCH CUSF FORMS EXIST.

<u>NB</u>: THE PROOF BY SELBERG AND IN GENERAL BY LANGLANDS OF THE ANALYTIC CONTINUATION OF EISENSTEIN SERIES IS CARRIED OUT FOR <u>ANY</u> DISCRETE GROUP AND MAKES <u>NO</u> USE OF ARITHMETIC. IT IS IMPORTANT THAT IT DOESN'T AS THIS ALLOWS FOR ALL THE ARITHMETIC APPLICATIONS!

- SELBERG PROVES A "WEYL LAW"

$$\#\{\lambda_j : \lambda_j \leq \lambda \text{ CUSP EIGENVAL}\} + M(\lambda) \sim \frac{\text{AREA}(\Gamma\backslash\mathbb{H})}{4\pi}\lambda$$

AS $\lambda \to \infty$.

HERE $M(\lambda)$ IS THE WINDING NUMBER OF THE CONSTANT TERM OF THE EISENSTEIN SERIES (NON-NEGATIVE).

FOR $PSL_2(\mathbb{Z})$, $M(\lambda)$ IS EXPRESSIBLE IN TERM OF $\zeta(s)$ $\Rightarrow$ $M(\lambda) = O(\sqrt{\lambda}\log\lambda)$

$\Rightarrow$ THAT MOST OF THE SPECTRUM IS CUSPIDAL!

SELBERG CONJECTURED THAT THE ABUNDANCE OF MAASS CUSP FORMS WAS TRUE IN GENERAL (DIMENSION 2 WEYL LAW).

IT TURNS OUT THAT THE OPPOSITE IS TRUE, THESE TRANSCENDENTAL FORMS ARE VERY UNSTABLE AND THEIR EXISTENCE IS TIED TO ARITHMETIC AND FUNCTORIALITY.

PHILLIPS - S :

$$X = \Gamma \backslash \mathbb{H}$$

$\mathcal{T}(X) = \mathcal{T}(\Gamma) :=$ THE DEFORMATION SPACE OF Γ IN THE SPACE OF LATTICES IN $SL_2(\mathbb{R})$

$\equiv$ DEFORMATION SPACE OF RIEMANN SURFACES WITH CUSPS.

THE CO-TANGENT SPACE TO $\mathcal{T}(\Gamma)$ AT Γ IS CANONICALLY THE SPACE OF <u>HOLOMORPHIC</u> QUADRATIC DIFFERENTIALS ON X ; OR WHAT IS THE SAME WEIGHT 4 CUSP FORMS (Q) FOR Γ.

IF u_j IS A MAASS CUSP FORM FOR Γ (Γ ARITHMETIC)

$$\lambda_j = \frac{1}{4} + t_j^2 \quad , \; t_j > 0$$

THEN IF THE RANKIN-SELBERG L-FUNCTION

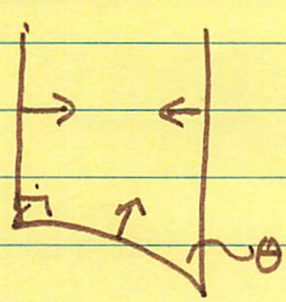
$$\boxed{L\left(\tfrac{1}{2} + i t_j, u_j \times Q\right) \neq 0}$$

THEN u_j IS DISSOLVED INTO A POLE OF THE EISENSTEIN SERIES ON DEFORMING Γ ALONG Γ_ε IN THE DIRECTION Q.

A "FERMI-GOLDEN RULE" GIVES AN EXACT DISSOLVING CONDITION IN TERMS ON THE NON-VANISHING OF A GLOBAL L-FUNCTION ON ITS CRITICAL LINE!

WOLPERT

TRIANGLE IN $\mathbb{H}$: T_θ



TRIANGLE WITH ANGLE θ
NEUMANN BOUNDARY CONDITIONS.
$\theta = \pi/3, \pi/6$ ARITHMETIC
HEJHAL NUMERICS
NO CUSP FORMS FOR ANY OTHER θ.

HILLIART-JUDGE (2018):

FOR ALL BUT COUNTABLY MANY θ, T_θ HAS NO L^2-EIGENFUNCTION WITH $\lambda > 0$.

- THERE IS NO ROBUST ANALYTIC CONSTRUCTION OF MAASS FORMS, IN THIS CASE AND MORE GENERALLY.

WHY DO WE CARE ABOUT THESE ELUSIVE OBJECTS FROM A NUMBER THEORETIC POINT OF VIEW?

TWO ILLUSTRATIVE EXAMPLES FOR GL_2

(A) $$d(n) := \sum_{v \mid n} 1$$ # OF DIVISORS OF n.

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s) \,;$$

THE "L-FUNCTION" OF A MAASS FORM NAMELY $\left.\frac{\partial E}{\partial s}\right|_{s=1/2} (z,s)$

CORRELATIONS OF $d(n)$ WITH ITS SHIFTS BY $h \geq 0$

$$\sum_{n \leq X} d(n)\, d(n+h)$$

AND ALSO ANALOGUES ON PROGRESSIONS.

SELBERG:

$$\sum_{n=1}^{\infty} \frac{d(n)d(n+h)}{n^s},$$ HAS A MEROMORPHIC CONTINUATION TO ALL OF $\mathbb{C}$

WITH POLES AT

$$\frac{1}{2} + it_j, \quad \lambda_j = \frac{1}{4} + t_j^2$$

MAASS CUSP FORMS FOR $SL_2(\mathbb{Z})$!

THIS ALSO MAKES CLEAR THE RELEVANCE IN APPLICATIONS OF THE "RAMANUJAN - SELBERG" CONJECTURE; THAT

$\lambda_1 \geq \frac{1}{4}$ FOR CONGRUENCE SUBGROUPS OF $SL_2(\mathbb{Z})$.

(B) HILBERT'S 11-TH PROBLEM

CONCERNING LOCAL TO GLOBAL PRINCIPLES FOR REPRENTATIONS OF <u>INTEGERS</u> IN A NUMBER FIELD BY INTEGRAL QUADRATIC FORMS:

- SIEGEL : HIS "MASS FORMULA" AND SIEGEL-WEIL FORMULA.

IN THE DEFINITE CASE THIS INVOLVES HOLOMORPHIC HILBERT MODULAR FORMS — VERY ALGEBRAIC IN NATURE, BUT POSSIBLY OF HALF INTEGRAL WEIGHT.

OVER $\mathbb{Q}$ SOLVED BY DUKE-IWANIEC
OVER K SOLVED BY COGDELL/PIATETSKY SHAPIRO/S.
⋮

- KEY INPUT ARE ALL THE MAASS FORMS!

- IN THESE AND RELATED APPLICATIONS OF ALL FORMS IT IS THE FULL SPECTRAL THEORY OF $L^2(\Gamma\backslash G)$ THAT CONTROLS THE DIOPHANTINE ANALYSIS VIA THE "RAMANUJAN CONJECTURES".

TRANSCENDENTAL NATURE AND GENERAL ROLE OF MAASS FORMS

$$G = GL_n \ , \quad n \geqslant 2 \ ; \ /\mathbb{Q}$$

OR K.

χ A UNITARY CENTRAL CHARACTER.

$$L^2_{cusp}\left(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}), \chi\right)$$

IS BY DEFINITION THE $GL_n(\mathbb{A})$ INVARIANT SUBSPACE OF ALL ϕ's FOR WHICH

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(ng)\, dn = 0 \ , \quad \text{FOR ALL } g \in G$$

WHERE N RANGES OVER ALL THE UNIPOTENT RADICALS OF THE $\mathbb{Q}$ PARABOLIC SUBGROUPS OF GL_n.

- NOT OBVIOUS THAT THIS SPACE IS BIG.
- L^2_{cusp} DECOMPOSES INTO A <u>DISCRETE</u> DIRECT SUM OF IRREDUCIBLE (UNITARY) REPRESETATIONS π OF $GL_n(\mathbb{A})$.

$$\pi \cong \bigotimes_v \pi_v$$

THERE COUNTABLY MANY SUCH π'S AND THEY ARE THE BUILDING BLOCKS (ATOMS) OF THE THEORY.

- THEY COME IN DIFFERENT FLAVORS DICTATED BY THEIR ARCHIMEDIAN COMPONENT (AND THIS IS SPECIAL TO NUMBER FIELDS) π_∞.

- COHOMOLOGICAL $(\mathfrak{g}, K)$ COHOMOLOGY
- MAASS TYPE.

AMONG THESE ARE ONES THAT CORRESPOND TO GALOIS REPRESENTATIONS

FINITE, ℓ-ADIC, ...
GREAT PROGRESS ON THESE, WILES/TAYLOR ...

BUT FOR HALF OF THE SAY THE FINITE 2-DIMENSIONAL ONES, THE EVEN ONES UNDER COMPLEX CONJUGATION WHICH ARE EXPECTED TO CORRESPOND TO MAASS FORMS WITH $\lambda = 1/4$; MUCH LESS PROGRESS.

TO UNDERSTAND THE NUMBER THEORETIC IMPORTANCE OF THESE VARIOUS SPECIES, WE SHOULD FIRST ASK HOW THEY ARE USED IN NUMBER THEORY; LEADS TO

"THE UNREASONABLE EFFECTIVENESS

OF AUTOMORPHIC FORMS IN NUMBER THEORY"

AMONG THE MANY THINGS THAT I WOULD POINT TO, IS THAT THEY ALLOW ONE TO UNDERSTAND THE VARIATION IN p OF LOCAL QUANTITIES.

THIS INCLUDES (BUT CERTAINLY NOT RESTRICTED TO) THE L-FUNCTIONS (EULER PRODUCTS)

$$L(s, \pi, \text{STANDARD})$$

WITH π AUTOMORPHIC CUSPIDAL.

IN FACT ALL (GOOD) EULER PRODUCTS ARE EXPECTED TO COME FROM THESE π's.

SO, IF YOUR INTEREST ARE L-FUNCTIONS GENERALIZING RIEMANN'S ZETA FUNCTION, THEN AT LEAST IN TERMS OF WHAT WE KNOW TODAY, YOU HAVE TO TURN TO THE GARDEN IN WHICH THEY ARE PLANTED: $\boxed{\left[L^2_{cusp} \right]}$.

I WOULD GO SO FAR AS TO SAY THAT FROM THIS POINT OF VIEW, IN THE FUNCTION FIELD (THAT IS $\mathbb{Q}$ OR K REPLACED BY $\mathbb{F}_q(t)$ OR A FINITE EXTENSION THEREOF) THE ROLE OF AUTOMORPHIC FORMS IS FAR LESS CENTRAL. AFTER ALL GROTHENDIECK'S COHOMOLOGY THEORY ALLOWS ONE TO PROVE THE CENTRAL THEOREMS ABOUT THE CORRESPONDING ZETA AND L-FUNCTIONS.

- NOTE THAT IN THE FUNCTION FIELD SETTING THERE IS NO ARCHIMEDIAN PLACE AND NO MAASS FORMS.

GL_1; HECKE CHARACTERS TRANSCENDENCE

IN FORMULATING CLASS FIELD THEORY

- WEIL DEFINED AN EXTENSION OF $GAL(\bar{K}/K)$, W_K, WHOSE 1-DIMENSIONAL REPRESENTATIONS CORRESPOND TO THE HECKE L-FUNCTIONS (GL_1 AUTOMORPHIC FORMS).

- HECKE DISTINGUISHED HIS CHARACTERS INTO TWO TYPES
  - FINITE IMAGE
  - INFINITE IMAGE "GROSSEN".

WEIL GOES FURTHER

- GROSSEN IS TYPE A_0 IF THEIR COEFF LIE IN A FIXED NUMBER FIELD.

- GROSSEN NOT OF TYPE A_0 WHEN THE VALUES ARE (PRESUMABLY) TRANSCENDENTAL.

INDEED THIS IS THE CASE AND FOLLOWS FROM THE SIX EXPONENTIALS THEOREM (LANG-SIEGEL, ... OBSERVED BY WALDSCHMIDT)

IF x_1, x_2, y_1, y_2, y_3 ARE LINEARLY INDEPENDENT OVER $\mathbb{Q}$ THEN AT LEAST ONE OF

$$e^{x_i y_j}$$

IS TRANSCENDENTAL.

THE ROLE OF THE TYPE A_0 CHARACTERS IN TERMS OF HASSE-WEIL ZETA FUNCTIONS WAS CLARIFIED BY WEIL, WHO ASKS ABOUT NON A_0.

- THEY ARE PART OF ANY ANALYSIS OF THE DISTRIBUTION PRIMES IN NUMBER FIELDS (HECKE). HECKE HAD TO INTRODUCE THEM EVEN IF ALL HE WANTED TO DO WAS ANALYTICALLY CONTINUE $\zeta_K(s)$!

FOR GL_n, $n \geq 2$

CLOZEL HAS FORMULATED DEFINTIONS OF π BEING TYPE A_0, BASICALLY IF IT'S SATAKE AND L-PARAMETERS LIE IN A FIXED NUMBER FIELD.
(BUZZARD-GEE AND MORE RECENTLY BERNSTEIN HAVE STUDIED THIS FURTHER AND POINTED TO SUBTLETIES IN THE DEFINITIONS CONNECTED WITH TWISTS).

WHERE DO MAASS FORMS LAND?

FOR $GL_2/\mathbb{Q}$, WE HAVE SEEN THAT THOSE WITH EIGENVALUE 1/4 PROBABLY ARE OF TYPE A_0 AND CORRESPOND TO FINITE TWO DIMENSIONAL GALOIS REPRESENTATIONS.

THE ONLY MODEST RESULT THAT I KNOW IS:

(S [illegible] BRUMLEY) IF ϕ IS A MAASS CUSP FORM WHOSE COEFFICIENTS ARE RATIONAL INTEGERS OR IN A QUADRATIC FIELD $\neq \mathbb{Q}(\sqrt{5})$ THE $\lambda = \lambda(\phi) = 1/4$ AND IT CORRESPONDS TO A TWO DIMENSIONAL SOLVABLE GALOIS REPRESENTATION.

• IF $\lambda > 1/4$ THEN I EXPECT (AND ASSUME SO DO OTHERS) THAT π HAS TRANSCENDENTAL PARAMETERS; THEY ARE THE ANALOGUE OF NOT TYPE A_0.

THE ONLY π'S FOR WHICH THIS IS KNOWN ARE THE ONES EXPICITLY CONSTRUCTED BY MAASS AS THETA LIFTS FROM $SO(1,1)$, IN WHICH CASE THE TRANSCENDENCE FOLLOWS FROM THE SIX EXPONENTIALS THM.

L_K ?

IF THE π'S IN L^2_{CUSP} ARE THE ATOMS THEN A FUNDAMENTAL QUESTION IS HOW DO THEY INTERACT.

ONE WAY IS MEASURED BY DETERMINING WHEN AN L-FUNCTION (OF THE GENERAL TYPE DEFINED BY LANGLANDS) THAT IS ASSOCIATED TO THEM HAS A POLE AT $S=1$.

FOR EXAMPLE IF $\pi_1, \dots, \pi_r$ ARE AS ABOVE WHEN DOES

$L(s, \pi_1 \times \cdots \times \pi_r)$ HAVE A POLE AT $s=1$?

IF THE π_j'S CORRESPOND TO GALOIS REPRESENTATIONS OR EVEN REPRESENTATIONS OF W_K THEN THE ANSWER IS GIVEN BY COMPUTING INVARIANTS.

HOW ABOUT ~~FOR~~ GENERAL π_j'S, MAASS FORMS.

LANGLANDS PROPOSES AN EXTENSION OF W_K CALLED L_K, WHOSE FINITE DIMENSIONAL REPRESENTATIONS CORRESPOND TO THE π_j'S AND WOULD EXPLAIN HOW THEY INTERACT.

I AM A BIT SKEPTICAL ABOUT A NON-CIRCULAR DEFINITION OF L_K FOR THE SIMPLE REASON THAT IT WOULD HAVE TO GIVE A FINITE DIMENSIONAL DESCRIPTION OF OUR ELUSIVE ϕ'S AT THE BEGINNING.

FUNDAMENTAL CONJECTURES AND COUNTING

THE MAIN CONJECTURES ABOUT THE π'S IN L^2_{cusp} ARE

(A) GENERALIZED RAMANUJAN CONJECTURE THAT:

π_v IS TEMPERED AT EVERY PLACE v.

(B) $L(s, \pi, \text{STANDARD})$ SATISIFIES THE RIEMANN HYPOTHESIS. ALL THE ZEROS OF THE COMPLETED L FUNCTION ARE ON $Re(s) = 1/2$.

WHILE (A) APPEARS TO BE SOLID WITH VERY STRONG APPROXIMATIONS TO ITS TRUTH BEING KNOWN; (B) IS MORE CONCERNING TO ME.

WE HIGHLIGHTED THE EXISTENCE AND ABUNDANCE OF π'S AS BEING CRITICAL TO THE THEORY. ACTUALLY FOR (B) IT IS CRITICAL THAT THERE NOT BE TOO MANY π'S.

THAT THERE ARE ONLY COUNTABLY MANY π'S IS GOOD NEWS (IT IS HARD TO IMAGINE (B) HOLDING IF THERE WERE CONTINUOUS DEFORMATIONS)

FOR A GIVEN π ONE CAN DEFINE THE ANALYTIC CONDUCTOR "$C(\pi)$" WHICH MEASURES THE COMPLEXITY OF π (IT IS MADE OUT THE RAMIFIED AND ARCHIMEDIAN PLACES).

GIVEN K AND n

$$\mathcal{F}_{K,n}(x) = \#\{\pi \in L^2_{cusp}(GL_n) : C(\pi) \leq x\}$$

IS FINITE.

BRUMLEY AND MILICEVIC (2020) HAVE ALL BUT PROVED A ~~SC~~ SCHANUEL-WEYL LAW:

$$\mathcal{F}_{K,n}(x) \sim c(\mathcal{F}_{K,n})\, x^{n+1} \quad \text{AS } x \to \infty,$$

WHERE $C(\mathcal{F}_{K,n})$ HAS A "TAMAGAWA NUMBER" INTERPRETATION.

NOW (B) IMPLIES A VERY STRONG SPACING BETWEEN DIFFERENT π'S WITH $C(\pi) \leq x$.

THAT IS π_v AND π'_v CANNOT BE CLOSE FOR $|v| \ll (\log x)^2$ WITHOUT π BEING EQUAL TO π'.

SO IT IS GOOD THAT $\mathcal{F}_{K,n}(x)$ IS LIMITED IN ITS GROWTH.

WHILE ALL SEEMS SAFE FOR n FIXED; MY CONCERN IS AS $n \to \infty$, COULD THERE BE TOO MANY π'S ?