Langlands correspondence and geometry

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drawing by M. Murphy

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Two primes, two abelian varieties

I'll begin by describing a problem that was posed to me by several mathematicians around the time that I moved in California (September, 1978). Let p and q be distinct prime numbers; for example 43 and 47.

Consider, on the one hand, the abelian variety $J_0(pq)$.

Consider, on the other hand, the Jacobian of the Shimura curve made from the group of norm-1 units in a maximal order of a rational quaternion algebra of discriminant pq. One might call this Jacobian $J_0^{pq}(1)$.

These two abelian varieties should be related. More precisely, one expects an isogeny between $J_0^{pq}(1)$ and an abelian subvariety of $J_0(pq)$.

“One expects”—why?

The cotangent space of $J_0^{pq}(1)$ is the space of weight-2 holomorphic forms on the norm-1 group from which $J_0^{pq}(1)$ was made. Similarly, $J_0(pq)$ corresponds to the space of weight-2 cusp forms on $\Gamma_0(pq)$.

In 1965, Hideo Shimizu found an amazing connection between the traces of Hecke operators T_n on the two spaces. For $n \geq 1$ the trace of T_n on the first space coincides with the trace of T_n on the *new part* of the space of weight-2 cusp forms on $\Gamma_0(pq)$.

Throw in the Eichler–Shimura relations and the Tate conjectures, and you see that there *should be* a Hecke-compatible isogeny

$$J_0^{pq}(1) \sim J_0(pq)^{\text{new}}.$$

Shimizu's article

H. Shimizu's result is regarded now as a special case of a Langlands correspondence, but it was a watershed contribution that launched a general theory. It concerned relations among quaternion algebras over a totally real field. Martin Eichler began his Math Reviews review of the article as follows:

> *The paper deals with quaternion algebras over a finite totally real algebraic number field. We report the principal result only in the case of quaternions over the rational field, however, since it is remarkable enough in this special case and its generalization is but a technical matter.*

I was surprised to discover that MathSciNet lists only 24 post-1979 references to the article.

Existence of the isogeny

In 1980, I used some of the techniques of Serre's “Abelian ℓ-adic Representations and Elliptic Curves” and results of my thesis to verify Tate's conjecture for endomorphisms of abelian varieties which are of $\mathbf{GL}(2)$ type and have no non-zero quotients with everywhere potential good reduction.

My article was entitled “Sur les variétés abéliennes à multiplications réelles”.

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Of course, my result was superseded by "Endlichkeitssätze für abelsche Varietäten über Zahlkörpern," a 1983 article by G. Faltings.

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An explicit isogeny?

In 1980, I considered that I had not responded fully to the isogeny question because I had not constructed a map (i.e., homomorphism of abelian varieties) between $J_0^{pq}(1)$ and anything related to $J_0(pq)$. Was I missing some correspondence on the product

$$X_0(1)^{pq} \times X_0(pq)?$$

The first factor classifies "fake elliptic curves"—abelian surfaces with an action of the maximal order in the rational quaternion algebra of discriminant pq. The second factor classifies `@realellipticcurves` with a cyclic subgroup of order pq.

How might the two types of objects be related? I couldn't discern any relation.

The "new part" of $J_0(pq)$

The "degeneracy" maps $X_0(pq) \rightrightarrows X_0(p)$ and $X_0(pq) \rightrightarrows X_0(q)$ give rise to a map (with finite kernel)

$$J_0(p) \times J_0(p) \times J_0(q) \times J_0(q) \xrightarrow{\alpha} J_0(pq),$$

whose image is the old subvariety $J_0(pq)_{\mathrm{old}}$ of $J_0(pq)$. The new quotient of $J_0(pq)$ is

$$J_0(pq)^{\mathrm{new}} := J_0(pq)/J_0(pq)_{\mathrm{old}}.$$

Similarly, the new subvariety $J_0(pq)_{\mathrm{new}}$ of $J_0(pq)$ is the connected component of the kernel of

$$J_0(pq) \longrightarrow J_0(p) \times J_0(p) \times J_0(q) \times J_0(q),$$

which is the dual of α.

The new "part" of $J_0(pq)$

The subvariety $J_0(pq)_{new}$ and the quotient $J_0(pq)^{new}$ are naturally dual to each other. Composing $J_0(pq)^{new} \hookrightarrow J_0(pq)$ with the quotient map $J_0(pq) \to J_0(pq)^{new}$ gives an isogeny

$$\iota : J_0(pq)_{new} \to J_0(pq)^{new}$$

with kernel $J_0(pq)_{new} \cap J_0(pq)_{old}$. Because $J_0(1)^{pq}$ is self-dual (being a Jacobian) and is isomorphic to a subquotient of $J_0(pq)$, one can imagine an isogeny

$$\lambda : J_0(pq)_{new} \to J_0(1)^{pq}$$

such that

$$\hat{\lambda} \circ \lambda : J_0(pq)_{new} \to J_0(pq)^{new}$$

is the isogeny ι. The kernel of λ will be a subgroup of $\Delta := J_0(pq)_{new} \cap J_0(pq)_{old}$, which suggests an analysis of Δ.

A CHaTty comment

I have always considered it unlikely that there could be natural maps

$$J_0(pq)_{new} \to J_0(1)^{pq}, \quad J_0(1)^{pq} \to J_0(pq)^{\mathrm{new}}$$

and yet no natural maps

$$J_0(pq) \to J_0(1)^{pq} \quad J_0(1)^{pq} \to J_0(pq).$$

Maybe it's time to reassess the situation?

Analysis of Δ

The group $\Delta = J_0(pq)_{new} \cap J_0(pq)_{old}$ can expressed simply as a subquotient of the kernel of the composite

$$J_0(p) \times J_0(p) \times J_0(q) \times J_0(q) \longrightarrow J_0(p) \times J_0(p) \times J_0(q) \times J_0(q)$$

and the kernel of the map

$$J_0(p) \times J_0(p) \times J_0(q) \times J_0(q) \to J_0(pq).$$

This latter kernel can be understood if one knows the kernels Σ_p and Σ_q of the two simpler maps

$$J_0(p) \times J_0(p) \to J_0(pq), \quad J_0(q) \times J_0(q) \to J_0(pq)$$

as well as the intersection inside $J_0(pq)$ of the images of these two maps.

Analysis of Δ

I computed the groups Σ_p and Σ_q in my 1983 ICM talk: they're the Shimura subgroups of $J_0(p)$ and $J_0(q)$, embedded anti-diagonally in the two products. That the kernel is no bigger than these groups amounts to "Ihara's lemma," which has become quasi-notorious in the arithmetic theory of reductive groups. My computation was used in *level raising*.

Level lowering is used to show that the two subvarieties $(J_0(p) \times J_0(p))/\Sigma_p$ and $(J_0(q) \times J_0(q))/\Sigma_q$ of $J_0(pq)$ intersect essentially not at all.

The result is a reasonably explicit understanding of the intersection Δ, which by the way is associated with Galois representations of level pq that are simultaneously old and new.

Subgroups of Δ

A question that might be worth exploring is whether there is a natural subgroup H of Δ so that the quotient $J_0(pq)_{\mathrm{new}}/H$ is auto-dual (and then a good candidate for being isomorphic to the Jacobian $J_0^{pq}(1)$).

An alternative point of view is to look for an isogeny $J_0(pq)^{\mathrm{new}} \to J_0(1)^{pq}$, i.e., a map $J_0(pq) \to J_0(1)^{pq}$ that is 0 on the old subvariety of $J_0(pq)$. Ogg has pointed out that $X_0(1)^{pq}$ has no cusps; therefore, the cuspidal subgroup of $J_0(pq)$ is likely to map to 0 in $J_0(1)^{pq}$ under any such map.

Some relevant literature

- Local diophantine properties of Shimura curves by Bruce W. Jordan and Ron A. Livne
- On the Néron model of Jacobians of Shimura curves by Bruce W. Jordan and Ron A. Livne
- On maps between modular Jacobians and Jacobians of Shimura curves by David Helm
- On Ribet's isogeny for $J_0(65)$ by Krzysztof Klosin and Mihran Papikian
- Galois extensions and a conjecture of Ogg by Krzysztof Klosin and Mihran Papikian

Multiplicity 1

Let $\mathbf{T}$ be the subring of $\operatorname{End} J_0(pq)$ generated by the Hecke operators T_n for $n \geq 1$. Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}$. Under mild hypotheses, $J_0(pq)[\mathfrak{m}]$ is a $\mathbf{T}/\mathfrak{m}$-vector space of dimension ≤ 2. (This is called "multiplicity 1.")

If multiplicity 1 holds for $J_0(pq)$, it holds a fortiori for the subvariety $J_0(pq)_{new}$.

Multiplicity 1

On the other hand $J_0^{pq}(1)[\mathfrak{m}]$ can be of dimension > 2. The failure of multiplicity 1 occurs in certain cases where the 2-dimensional mod $\mathfrak{m}$ representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ associated to $\mathfrak{m}$ is irreducible and simultaneously old and new. Specifically, suppose that the residue characteristic of $\mathfrak{m}$ is different from 2, p and q. The failure comes about only when the Galois representation is unramified at one of the two primes p, q; to fix ideas, say it's unramified at p. Failure then occurs precisely when the Frobenius element for p in $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ operates as a scalar (± 1) in the representation. This is an unusual situation—but it occurs.

If multiplicity 1 fails for $J_0^{pq}(1)[\mathfrak{m}]$ but holds for $J_0(pq)$, the kernel of every isogeny $J_0(pq)_{new} \to J_0^{pq}(1)$ necessarily involves $\mathfrak{m}$. This is a fact to keep in mind as one tries to guess the relationship between the two abelian varieties.

A second CHaTty comment

A few slides ago, I raised the possibility of identifying a finite subgroup H of $\Delta = J_0(pq)_{\text{old}} \cap J_0(pq)_{\text{new}}$ so that $J_0(pq)_{new}/H$ is isomorphic to $J_0(1)^{pq}$.

In light of the multiplicity 1 comments, it might be worth considering the following subgroups (and their analogues with the roles of p and q transposed):

- The group of points in Δ that are fixed by all decomposition groups for p in $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. (In other words, these points are unramified at p and fixed by all Frobenius elements for p in $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.)
- The group of points in Δ that are unramified at p and sent to their negatives by all Frobenius elements for p in $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

Passing to characteristic p

The final idea that I wish to explore goes as follows: Let's suppose that there is a natural map $J_0(pq) \to J_0^{pq}(1)$ over $\mathbf{Q}$. Then this map induces a map between the Néron models of these two abelian varieties and then between the mod p reductions of the two Néron models. (We could speak loosely of the mod p reductions of $J_0(pq)$ and $J_0^{pq}(1)$, but these two varieties have bad reduction at p unless p is very small.)

In characteristic p, both Néron models contain a maximal torus (i.e., multiplicative-type group), and one gets a map between the two tori.

The two tori are known explicitly (Deligne–Rapoport and Cherednik–Drinfeld, respectively), so we'd end up with a map between explicit objects. What could that be?

Tori

The spoiler here is that I never found a map between the tori, or between the character groups of the two tori—which is the same thing.

The problem is that the arithmetic of the torus for $J_0(pq)$ in characteristic p is controlled by the rational quaternion algebra of discriminant p, whereas the arithmetic for $J_0(1)^{pq}$ is controlled by the rational quaternion algebra of discriminant q. This seems really bad.

Straw into gold

Somehow the character group for the reduction of $J_0(pq)$ in characteristic p is closely related to the character group for the reduction of $J_0(1)^{pq}$ in characteristic q.

This is amazing and counterintuitive. It can actually be interpreted as a canonical bijection between isomorphism classes of certain objects over $\overline{\mathbf{F}}_p$ and isomorphism classes of other objects over $\overline{\mathbf{F}}_q$. The only wrinkle is that one has to be careful in choosing the algebraic closures $\overline{\mathbf{F}}_p$ and $\overline{\mathbf{F}}_q$: begin with a maximal order in a quaternion algebra of discriminant pq. This ring has residue fields isomorphic to $\mathbf{F}_{p^2}$ and $\mathbf{F}_{q^2}$. The fields $\overline{\mathbf{F}}_p$ and $\overline{\mathbf{F}}_q$ need to be taken as algebraic closures of the finite fields $\mathbf{F}_{p^2}$ and $\mathbf{F}_{q^2}$.

Straw into gold

In 1986, the relations between the character groups for $J_0(1)^{pq}$ in characteristic q, and $J_0(pq)$ in characteristic p formed the key "new" ingredient of my proof of Serre's level-lowering conjecture—the "Conjecture ϵ" that was fingered by Serre in 1985 as the sole obstacle to validating Frey's implication

Modularity of elliptic curves over $\mathbf{Q} \Longrightarrow$ Fermat's Last Theorem.