The conjectures of Gan, Gross, and Prasad

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Wee Teck Gan and Dipendra Prasad

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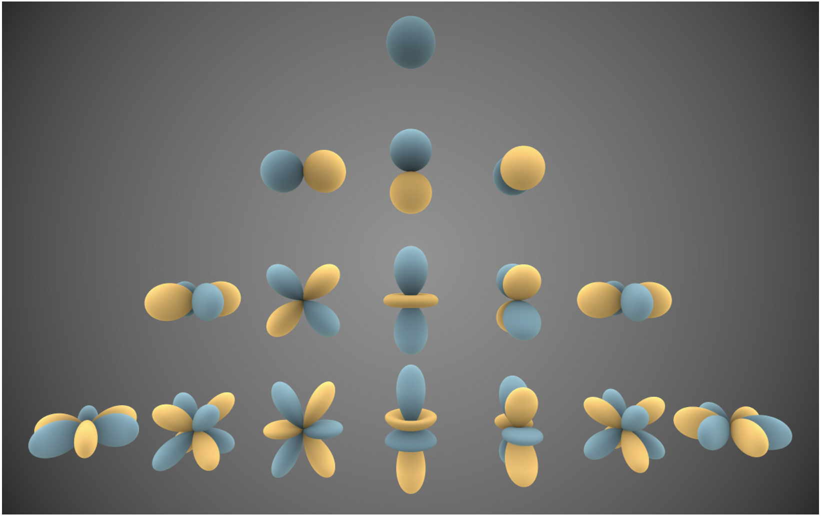
For the precise conjectures, see volume 346 of Astérisque.



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Let W_ℓ be the vector space of homogeneous polynomials $f(x, y, z)$ degree ℓ which are harmonic on $\mathbb{R}^3$

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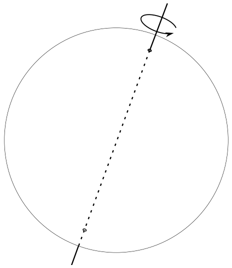
Then W_ℓ is an irreducible representation of $\mathrm{SO}(3)$ of dimension $2\ell + 1$, and

$$\mathscr{F}(S^2) = \widehat{\bigoplus}_{\ell \geq 0} W_\ell$$

The subgroup of SO(3) which fixes a point on S^2 is isomorphic to the rotation group SO(2)



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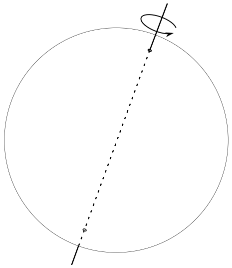

The restriction of W_ℓ decomposes as a sum of one-dimensional representations

$$\mathrm{Res}_{\mathrm{SO}(2)} W_\ell = \bigoplus_{|m| \leq \ell} \chi_m \qquad \chi_m(z) = z^m.$$

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The restriction is given by the branching formula

$$\mathrm{Res}_{\mathrm{SO}(2n)} W_\alpha = \bigoplus U_\beta \qquad \alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \ldots \alpha_n > |\beta_n|.$$

Since the restriction is multiplicity-free

$$\dim \mathrm{Hom}_{\mathrm{SO}(2n)}(U, W) = \dim \mathrm{Hom}_{\mathrm{SO}(2n)}(W \otimes U^\vee, \mathbb{C}) \leq 1$$

for all irreducibles W of $\mathrm{SO}(2n+1)$ and U of $\mathrm{SO}(2n)$.

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For every irreducible representation W of the special orthogonal group $\mathrm{SO}_{2n+1}(k)$, where k is a local field, the restriction to the subgroup $\mathrm{SO}_{2n}(k)$ is **multiplicity-free**.

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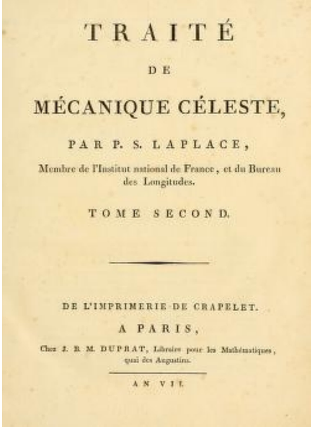
The local conjecture addresses the question: what is the corresponding branching formula?

To answer that question, we need to leave the harmonic analysis in Pierre Laplace's Méchanique Céleste,



TRAITÉ

DE

MÉCANIQUE CÉLESTE,

PAR P. S. LAPLACE,

Membre de l'Institut national de France, et du Bureau des Longitudes.

TOME SECOND.

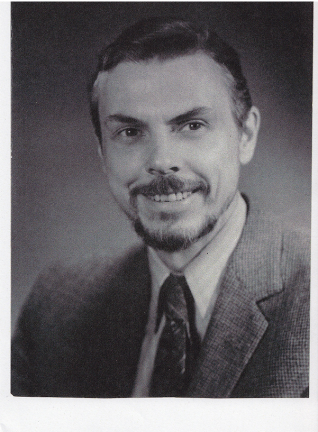
DE L'IMPRIMERIE DE CRAPELET.

A PARIS,

Chez J. B. M. DUPRAT, Libraire pour les Mathématiques, quai des Augustins.

AN VII.

and turn to the harmonic analysis in John Tate's PhD thesis.



Fourier Analysis in Number Fields and Hecke's Zeta-Functions†

J. T. TATE

ABSTRACT

We lay the foundations for abstract analysis in the groups of valuation vectors and idèles associated with a number field. This allows us to replace the classical notion of ζ-function, as the sum over integral ideals of a certain type of ideal character, by the corresponding notion for idèles, namely, the integral over the idèle group of a rather general weight function times an idèle character which is trivial on field elements. The role of Hecke's complicated theta-formulas for theta functions formed over a lattice in the n-dimensional space of classical number theory can be played by a simple Poisson formula

Hecke's zeta functions generalize the Riemann zeta function, which satisfies the functional equation

$$\zeta^*(s) = (\pi)^{-s/2}\Gamma(s/2)\prod(1-p^{-s})^{-1} = \zeta^*(1-s)$$

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A Hecke character is a continuous homomorphism

$$\chi = \prod \chi_v : \mathbb{A}^*/k^* \longrightarrow \mathbb{C}^*$$

The L-function of χ is defined as an Euler product of local terms

$$L(\chi, s) = \prod L(\chi_v, s)$$

which converges in a right half plane.

Tate gave a new proof of Hecke's analytic continuation and functional equation

$$L(\chi, s) = \epsilon(\chi)\, A(\chi)^{1/2-s}\, L(\bar{\chi}, 1-s)$$

and factored the constant $\epsilon(\chi)$ into local terms

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$$W(\mathbb{C}) = \mathbb{C}^* \quad W(\mathbb{R}) = N(\mathbb{C}^*) \subset \mathbb{H}^* \quad W(k_v) \subset \mathrm{Gal}(\bar{k}_v/k_v).$$

Deligne defined local epsilon factors for higher dimensional representations M_v of the Weil group, which satisfy

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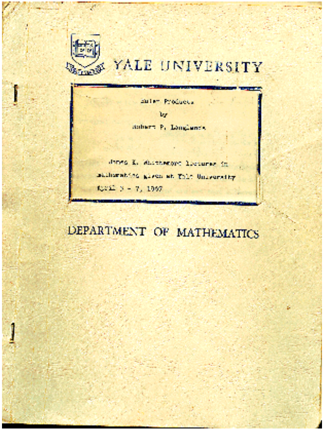
The local GGP conjecture relates the **branching laws** from $\mathrm{SO}_{2n+1}(k_v)$ to $\mathrm{SO}_{2n}(k_v)$ to the signs of **symplectic epsilon factors**.

We will use a bridge between representation theory and number theory which was constructed by Robert Langlands.



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The parameter of the irreducible representation W_α of $\mathrm{SO}(2n+1)$ is the symplectic representation

$$M = J(\alpha_1) \oplus J(\alpha_2) \oplus \ldots \oplus J(\alpha_n)$$

and the parameter of the irreducible representation U_β of $\mathrm{SO}(2n)$ is the orthogonal representation

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In fact, the representations M and N of $W(k_v)$ parametrize a **finite set** of irreducible representations, called an L-packet.

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The local GGP conjecture states that there is a **unique** irreducible representation $W \otimes U$ in each **generic** L-packet with

$$\dim \mathrm{Hom}_{\mathrm{SO}_{2n}}(W \otimes U, \mathbb{C}) = 1$$

whose character is given by

$$\psi(a, 1) = \epsilon(M^{a=-1} \otimes N) \times \det N(-1)^{\dim M^{a=-1}/2}$$

$$\psi(1, b) = \epsilon(M \otimes N^{b=-1}) \times \det N^{b=-1}(-1)^{\dim M/2}$$

In particular, on the center of $\mathrm{Sp}(M) \times \mathrm{SO}(N)$:

$$\psi(-1, 1) = \psi(1, -1) = \epsilon(M \otimes N) \det N(-1)^{\dim M/2}.$$

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We now turn to the global conjecture.

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An obvious necessary condition is

$$\mathrm{Hom}_{H(\mathbb{A})}(W \otimes U, \mathbb{C}) = \prod \dim_{H(k_v)}(W_v \otimes U_v, \mathbb{C}) \neq 0$$

so the local components are distinguished in their L-packets.

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Hence $L(M \otimes N, s)$ vanishes to even order at the point $s = 1/2$.

The Hasse-Witt invariant of the local orthogonal spaces, for the groups acting on $W_v \otimes U_v$ is related to

$$\epsilon_v(M \otimes N). \det N_v(-1)^{\dim M_v/2}.$$

Since we have global orthogonal spaces, Hilbert's reciprocity law implies that

$$\prod \epsilon_v(M \otimes N) = +1$$

This is the sign $\epsilon(M \otimes N)$ in the functional equation of the tensor product L-function

$$L(M \otimes N, s) = \prod_v L_v(M \otimes N, s).$$

Hence $L(M \otimes N, s)$ vanishes to even order at the point $s = 1/2$.

The global conjecture predicts that the diagonal period integral is non-zero on $W \otimes U$ if and only if

$$L(M \otimes N, 1/2) \neq 0.$$

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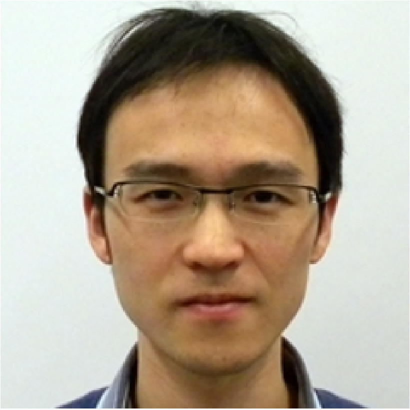

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One can make such a cycle from the diagonally embedding Shimura variety $T = T_{2n-2}$. The height pairing $\langle T, * \rangle$ on the Chow group gives an $\mathrm{SO}_{2n}(\mathbb{A}^f)$ invariant linear form.

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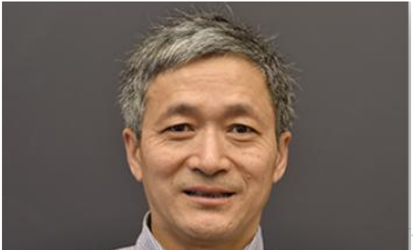

When $n = 1$, $k = \mathbb{Q}$, and the group $\mathrm{SO}_3(\mathbb{A}^f)$ is split, S is the modular curve $X_0(N)$ and the cycle T is given by Heegner points, which are rational over the Hilbert class field of $K = \mathbb{Q}(\sqrt{-D})$.

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When P has infinite order, Victor Kolyvagin proved that $E(K)$ has rank 1.

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Inventiones mathematicae

Heegner points and derivatives of L-series

Benedict H. Gross[1] and Don B. Zagier[2]

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to John Tate

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Compositio Mathematica 54 (1985) 173–242.

SUR LES VALEURS DE CERTAINES FONCTIONS L AUTOMORPHES EN LEUR CENTRE DE SYMETRIE

J-L. Waldspurger

Il y a quelques années, Vignéras a démontré le résultat suivant. Soit f une forme modulaire holomorphe parabolique de poids k pair, de caractère trivial, pour un groupe de congruence $\Gamma_0(N)$. On suppose que f est une newform. Pour un nombre premier p, soit a_p la valeur propre de l'opérateur de Hecke T_p associée à f. Notons $\mathbb{Q}(f)$ le sous-corps de $\mathbb{C}$ engendré par les a_p. C'est une extension finie de $\mathbb{Q}$. Soient χ un caractère de Dirichlet quadratique, de conducteur premier à N, et tel que $\chi(-1)=1$, et f' la newform telle que, pour presque tout p, f' soit propre pour l'opérateur de Hecke T_p, de valeur propre $\chi(p)a_p$. Notons $L(f, s)$, $L(f', s)$, les fonctions L habituelles associées à f et f', supposons $L(f, k/2) \neq 0$, $L(f', k/2) \neq 0$. Alors, à un facteur explicite près, le rapport $L(f', k/2)L(f, k/2)^{-1}$ est le carré d'un élément de $\mathbb{Q}(f)$ ([V]). Pour démontrer ce résultat, Vignéras exprimait ces valeurs de fonctions L en termes the coefficients de Fourier de formes modulaires de poids demi-entier. On démontre ici ce même résultat, sous une forme plus générale, par une méthode tout-à-fait différente.

Soient F un corps de nombres, M une algèbre de quaternions définie sur F, G le groupe de ses éléments inversibles, E' un sousmodule irréductible de l'espace des formes automorphes paraboliques sur $G(F)\backslash G(\mathbb{A})$ (cf. ci-dessous *notations* et II,1), π' la représentation automorphe de $G(\mathbb{A})$ dans E', ω son caractère central, π la représentation automorphe de $\mathrm{GL}_2(\mathbb{A})$ associée à π' par la correspondance de Jacquet-Langlands, T un sous-tore maximal de G défini sur F, F_T l'extension quadratique de F associée à T, Ω un caractère de $T(F)\backslash T(\mathbb{A})$ coïncidant avec ω sur le centre $Z(\mathbb{A})$ de $G(\mathbb{A})$, Π la représentation automorphe de $\mathrm{GL}_2(F_T(\mathbb{A}))$ qui relève π (cf. *notations*). On peut considérer Ω comme un caractère de $F_T^{\times}(\mathbb{A})$, et définir la fonction $L(\Pi \otimes \Omega^{-1}, s)$. Soit $e' \in E'$, considérons l'intégrale

$$\int_{T(F)Z(\mathbb{A})\backslash T(\mathbb{A})} e'(t)\Omega^{-1}(t)\mathrm{d}t.$$

Le point fondamental est de montrer que (grosso modo) le carré de cette intégrale est égale au produit de trois termes: un terme indépendant de T

For more information on what happened next, see

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Thank you!