Codimension one in Algebraic and Arithmetic Geometry.

Hélène Esnault
FU Berlin/Harvard/Copenhagen

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equivalently

$\Longrightarrow$?? $[H^i_{dR}(X) \xrightarrow{\text{rest}=0} H^i_{dR}(\mathbb{C}(X)) := \lim_U H^i_{dR}(U)]$

ℓ-adic analog

weights

The notions of weight in complex geometry and in ℓ-adic theory in geometry over a finite field have been developed by Deligne and by the Grothendieck school. The analogy between the theories is foundational and led to predictions and theorems on both sides.

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Hodge filtration

On the complex Hodge theory side, not only do we have the weight filtration, but we also have the Hodge filtration. The analogy on the ℓ-adic side of the Hodge filtration over a finite field hasn't really been documented (by Deligne and by no-one).

ℓ-adic analogies

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more generally: ℓ-adic analog of the Hodge filtration?

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CHAT's motto: the narrative of a theorem

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Let me explain why and give some prospective.

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Algebraicity

so remarkable fact

For this particular case $H^i(X, \mathcal{O}) = 0$ making $H = H^i_{dR}(X)$ of Hodge type $(i-1,1), \ldots, (1, i-1)$, the codimension 1 conjecture is expressible over the field of definition $K \subset \mathbb{C}$ of X.

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codimension 1 conjecture in this case is purely algebraic

It says the relation is iff in $(\star)$:

$H^i(X,\mathcal{O}) = 0 \Longrightarrow ??\ [H^i_{dR}(X) \xrightarrow{\text{rest}=0} H^i_{dR}(K(X))]$

Examples

1st example

$i = 1$: $H^1(X, \mathcal{O}) = 0 \iff H^1_{dR}(X) = 0$ by Hodge theory: indeed $0 \to H^0(X, \Omega^1) \to H^1_{dR}(X) \to H^1(X, \mathcal{O}) \to 0$ (Hodge-to-de Rham-degeneration) plus Hodge duality $h^{10} = h^{01}$.

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2nd example

$i = 2$: $H^2(X, \mathcal{O}) = 0 \iff H^2_{dR}(X) = NS(X) \otimes \mathbb{Q}$ by Hodge theory: indeed
exponential sequence $1 \to \mathbb{Z}(2\pi\sqrt{-1}) \to \mathcal{O}_{\mathrm{an}} \xrightarrow{\exp} \mathcal{O}^\times_{\mathrm{an}} \to 0$
$\implies H^1(\mathcal{O}^\times_{\mathrm{an}}) \xrightarrow{\mathrm{surj}} H^2(X, \mathbb{Z}(2\pi\sqrt{-1}))$ and GAGA $\implies$
$H^1(\mathcal{O}^\times_{\mathrm{an}}) = H^1(\mathcal{O}^\times) = \mathrm{Pic}(X)$.

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So purely analytic proofs in spite of purely algebraically formulated problem; those are the *only known* examples.

Motivic implication

Bloch's decomposition of the diagonal

$$CH_0(X_{\mathbb{C}}) = \mathbb{Q} \implies \forall i \geq 1 \qquad [H^i_{dR}(X) \xrightarrow{\text{rest}=0} H^i_{dR}(K(X))]$$

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In fact the motivic conjectures imply iff:
$H^i(X, \mathcal{O}) = 0 \ \forall i \geq 1 \implies CH_0(X_{\mathbb{C}}) = \mathbb{Q}$ so in particular it implies $[H^i_{dR}(X) \xrightarrow{\text{rest}=0} H^i_{dR}(K(X))] \ \forall i \geq 1.$

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$CH_0(X_{\mathbb{C}}) = \mathbb{Q} \implies \forall i \geq 1 \quad H^i(X, \mathcal{O}) = 0.$

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In fact the motivic conjectures imply iff:
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Truly this is very bold and might make us dizzy.

Narrative

I was driving

on the highway between Essen (Germany) and Paris (France). It was raining, the windscreen wipers were scratching on the window with a regular squeaky noise. All of a sudden in Belgium I thought on a possible analogy to the condition $H^i(X, \mathcal{O}) = 0$ *for* X *smooth projective defined over* $\mathbb{F}_q$*. Let me first explain why one wants the analogy and then what it is.*

Integrality

Deligne's integrality

X smooth $/\mathbb{F}_q \implies$ the eigenvalues of the geometric Frobenius acting on $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Z}_\ell)$ lie in $\bar{\mathbb{Z}}$.

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Gabber's purity together with localization

$\Longrightarrow$: if $[H^i(X_{\mathbb{F}_p}, \mathbb{Q}_\ell) \xrightarrow{\text{rest } =0} H^i(\mathbb{F}_p(X_{\mathbb{F}_p}), \mathbb{Q}_\ell)]$, then the eigenvalues of the geometric Frobenius acting on $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Z}_\ell)$ $\forall i > 0$ all lie in $q \cdot \bar{\mathbb{Z}}$.

Divisibility of the eigenvalues of the geometric Frobenius

Grothendieck-Lefschetz Trace Formula

$|X(\mathbb{F}_q)| = 1 + \sum_{i\ge 1}(-1)^i \mathrm{Tr\ Frob}|H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Z}_\ell)$

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$[H^i(X_{\mathbb{F}_p}, \mathbb{Q}_\ell) \xrightarrow{\text{rest } =0} H^i(\mathbb{F}_p(X_{\mathbb{F}_p}), \mathbb{Q}_\ell)] \Longrightarrow |X(\mathbb{F}_q)| \equiv 1 \bmod q.$

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So on the highway

it vaguely appeared to me that is one has an analog to $H^i(X, \mathcal{O}) = 0 \ \forall i \geq 1$ *then one would wish to have* $[H^i(X_{\mathbb{F}_p}, \mathbb{Q}_\ell) \xrightarrow{\text{rest} \ =0} H^i(\mathbb{F}_p(X_{\mathbb{F}_p}), \mathbb{Q}_\ell)] \ \forall i \geq 1$ *and then one would obtain* **not only the existence of a rational point on X but in fact even a congruence for the number of rational points**.

Wishfully

The analog of $H^i(X, \mathcal{O}) = 0$

for X smooth projective over $\mathbb{F}_q$ is: the eigenvalues of the geometric Frobenius acting on $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Z}_\ell)$ all lie in $q \cdot \bar{\mathbb{Z}}$

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while obviously the analog of $[H^i_{dR}(X) = 0 \xrightarrow{\text{rest } =0} H^i_{dR}(K(X))]$

for X smooth projective over $\mathbb{F}_q$ is:
$[H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell) \xrightarrow{\text{rest } =0} H^i(\bar{\mathbb{F}}_p(X_{\mathbb{F}_p}), \mathbb{Q}_\ell)]$

So for the Lang-Manin conjecture

Has to see

Bloch's (and later Bloch-Srinivas') technic shows as well: X smooth projective over $\mathbb{F}_q$ then

$$CH_0(X_{\overline{\mathbb{F}_q(X)}}) = \mathbb{Q} \Longrightarrow [H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell) \xrightarrow{\text{rest } =0} H^i(\bar{\mathbb{F}}_p(X_{\mathbb{F}_q}), \mathbb{Q}_\ell)]$$
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as, essentially by definition

X smooth projective over $\mathbb{F}_q$, X rationally connected
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Theorem (generalization of the Lang-Manin conjecture) (2002)

$$CH_0(X_{\overline{\mathbb{F}_q(X)}}) = \mathbb{Q} \Longrightarrow |X(\mathbb{F}_q)| \equiv 1 \bmod q$$

Analogy also yields formulation of

"Tate conjecture" in codimension 1

X smooth projective over $\mathbb{F}_q$ such that the eigenvalues of the geometric Frobenius on $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ lies in $q \cdot \bar{\mathbb{Z}} \Longrightarrow$??

$[H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell) \xrightarrow{\text{rest } =0} H^i(\bar{\mathbb{F}}_p(X_{\mathbb{F}_q}), \mathbb{Q}_\ell)]$

Away from analogies: p-adic Hodge theory

Grothendieck's codimension 1 conjecture

seen on $X_{\bar{\mathbb{Q}}_q}$, in étale p-adic cohomology

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for p large.

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prismatic lift

Those two cohomologies are induced from prismatic cohomology on $X_{\mathbb{Z}_p}$. It is natural to ask whether this prismatic lift yields a non-trivial information on the problem.